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Algebra Preliminary Examination August 2000

1) Let A, B be $n \times n$ complex matrices. If $A = (a_{ij})$, then $tr(A) = \sum_{i=1}^{n} a_{ii}$.

We call $tr(A)$ the trace of A.

(i) Show that $tr(AB) = tr(BA)$.

(ii) Show that similar matrices have the same trace.

(iii) Show that a nilpotent matrix has trace 0.

2) Let R be a commutative ring with 1 and M a right R-module. Suppose that $f : M \longrightarrow M$ is an R-module homomorphism with the property that $f^2 = f$. Show that $M = Ker(f) \oplus f(M)$.

3) (a) For which values of n are all Abelian groups of order n cyclic?

(b) Show that a group of order 12 has a normal Sylow subgroup.

4) Let D be a principal ideal domain.

(a) Show that every prime ideal of D is maximal.

(b) If $D[x]$ is also a principal ideal domain, show that D is a field.

5) Let C be the center of the group G. If C has index n in G, show that every conjugacy class in G has at most n elements. (If $a \in G$, the conjugacy class of a is $\{gag^{-1} \mid g \in G\}$.)

6) For each of the following, tell if it is true or false and give a reason.

i) If ϕ is an onto homomorphism from the group Z to the infinite group G, then ϕ is an isomorphism.

(ii) If M is a Q-module (Q is the rationals) and N is a non-zero submodule, then N is a free Q-module.

(iii) The subring of the rationals given by $\{m/n \mid m, n \in Z,\ n \text{ odd}\}$ has a unique maximal ideal.

(iv) Suppose G_1 and G_2 are finite groups and that H_i is a normal subgroup of G_i, for $i = 1, 2$. If $H_1 \simeq H_2$ and $G_1/H_1 \simeq G_2/H_2$, then $G_1 \simeq G_2$.