MATH313–Preliminary Examination.

August 2000

Instructions: Answer three out of the four questions. *You do not have to prove results which you rely upon, just state them clearly !*

Q1) (a) Prove: A qudrature formula $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ that uses the $n+1$ distinct nodes $x_0, \ldots, x_n$ and is exact of order at least n is interpolatory, that is,

$$\alpha_k = \int_a^b L_k(x)dx, \quad k = 0, \ldots, n,$$

where

$$L_k(x) = \frac{\prod_{\substack{j=0 \\ j\neq k}}^{n} (x - x_j)}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j)}, \quad k = 0, \ldots, n.$$

(b) The Legendre polynomial of degree n is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n,$$

with $P_0(x) \equiv 1$. Prove (verify) that for $k = 0, 1, \ldots, n-1$,

$$\int_{-1}^{1} x^k P_n(x)dx = 0.$$

Q2) (a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, \ldots.$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic (that is, the leading coefficient is 1).

(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty \geq 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm on the interval $[-1, 1]$.

(c) Let $\mathcal{S}$ be the subspace of $C[a, b]$ spanned by $\{1, x, x^2, \ldots, x^{n-1}\}$. Define $\text{dist}(x^n, \mathcal{S}) = \inf_{p\in\mathcal{S}} (\|7x^n - p\|_\infty)$. Show that $\text{dist}(x^n, \mathcal{S}) = 7(b-a)^n/2^{2n-1}$.

Q3) a) Let $x = (x_1, \ldots, x_n)^T$ be a vector whose entries are all **positive** numbers and for any vector $y \in \mathbb{R}^n$, define the quantity

$$f(y) \ := \ \inf\{\alpha > 0 \mid -\alpha x \leq y \leq \alpha x\},$$

where for two vectors $u, w \in \mathbb{R}^n$, $u \leq w$ means the every entry in w is at least equal to the corresponding entry in v. Show that $f(y)$ defines a vector norm on R^n. In the case that $x = (1, \ldots, 1)^T$, can you identify the common norm which now $f(y)$ yields?

b) Let $A \in \mathbb{C}^{n,n}$ and let

$$\rho(A) \ := \ \max\{|\lambda| \mid \det(A - \lambda I) = 0\}.$$

Show that the following statements are equivalent:

i) $\lim_{i\to\infty} A^i \ = \ 0.$

ii) $\rho(A) \ < \ 1.$

iii) There exists a multiplicative matrix norm $\|\cdot\|$ such that in this norm, $\|A\| \ < \ 1.$

c) Suppose that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that there is no multiplicative norm $\|\cdot\|$ for which $\|A\| = 1$. Hence find an example of a nonmultiplicative norm.

Q4) a) Show that if $B = (b_{i,j})$ is an **invertible** $n \times n$ **lower (upper) triangular matrix**, then B^{-1} is an $n \times n$ lower (upper) tringular matrix.

Recall now that the LU–factorization of A, $A = LU$, where L is a lower tringular matrix and U is an upper triangular matrix, is called **normalized** if the diagonal entries of L are all 1's.

Use the intial part of the question to show that a normalized LU–factorization of a nonsingular matrix A is unique, namely, if $A = LU = L'U'$ are normalized LU–factorizations of A, then $L = L'$ and $U = U'$.