Summer 00

PhD Prelim Exam — Measure & Integration

You can assume all measure spaces to be σ–finite.

1. Prove or disprove:

i Every Riemann–integrable function on $[0,\infty)$ is Lebesgue–integrable.

ii Every positive Riemann–integrable function on $[0,\infty)$ is Lebesgue–integrable.

iii Every Lebesgue–integrable function on $[0,\infty)$ is Riemann–integrable.

(Note: a function f is Riemann–integrable on $[0,\infty)$ means that its improper Riemann integral exists and is finite.)

2. Suppose $(X,\mathfrak{A},\mu)$ is a finite measure space. Prove or disprove:

i Every sequence of $\mathfrak{A}$–measurable functions that converges in the $L^1(\mu)$–norm converges a.e. (μ).

ii Every sequence of $\mathfrak{A}$–measurable functions that converges a.e. (μ) converges in measure (μ).

iii Every sequence of $\mathfrak{A}$–measurable functions that converges in the $L^1(\mu)$–norm converges in measure.

3.

i Prove that if $f \in L^1(\mu)$, then

$$\lambda(E) = \int_E f \, d\mu, \qquad E \in \mathfrak{A}$$

defines a $\mathbb{C}$–measure on $(X,\mathfrak{A})$.

ii What is the Radon–Nikodym derivative of λ with respect to μ?

iii Prove from first principles (without invoking any "big–name" theorems) that if $\lambda(E) = 0$ for all $E \in \mathfrak{A}$, then $f = 0$ a.e. (μ).

4.

i Prove the Generalized Minkowski inequality: If $(X,\mathfrak{A},\mu)$ and $(Y,\mathfrak{B},\lambda)$ are measure spaces, f a $(\mathfrak{A}\times\mathfrak{B})$–measurable function on $X\times Y$, and $p \in [1,\infty)$, then

$$\left(\int_X \left(\int_Y |f(x,y)|\ \lambda(dy)\right)^p \mu(dx)\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X |f(x,y)|^p\ \mu(dx)\right)^{\frac{1}{p}} \lambda(dy).$$

ii Explain how the inequality above generalizes the usual Minkowski inequality.

5. Suppose f is a $\mathbb{C}$–valued measurable function on a measure space (X,μ).

i Prove that if $\|f\|_{L^r} < \infty$ for some $r < \infty$, then

$$\|f\|_{L^p} \to \|f\|_{L^\infty} \quad \text{as } p \to \infty.$$

ii Prove that if $\mu(X)$, and $\|f\|_{L^r} < \infty$ for some $r > 0$, then

$$\lim_{p\to 0} \|f\|_{L^p} = \exp\left\{\int_X \log|f|\ d\mu\right\}.$$

(Define $\exp(-\infty) = 0$.)

6.

i Suppose (f_n) is a sequence of functions in $L^p(X,\mu)$, $p \in (1,\infty)$, which converges a.e. (μ) to $f \in L^p(X,\mu)$. Prove that if $\|f_n\|_{L^p} \leq 1$ for all n, then for all $g \in L^q(X,\mu)$ $(\frac{1}{p} + \frac{1}{q} = 1)$

$$\int_X f_n\, g\, d\mu \to \int_X f\, g\, d\mu \quad \text{as } n \to \infty.$$

ii Does your proof (of the statement in i) extend to $p = 1$ and $q = \infty$? If it does, then show how; if it does not, then show where the proof fails, and produce a counter–example to i in this case.