Preliminary Examination

Complex Analysis

2001

Instructions: Do all problems. Show your work in order to receive **ANY** credit.

Problem 1: Show that the function $u(z) := \log|z|$ has no harmonic conjugate in the domain $D = \mathbb{C} \setminus \{0\}$.

Problem 2: Evaluate

$$\int_{|z|=2} \frac{e^{\pi z}}{z^3 + z}\, dz.$$

Problem 3: Suppose f is analytic in the unit disk Δ and $f(0) = f'(0) = 0$. Also assume that $|f'(z)| \leq 1$ for all $z \in \Delta$. Prove that

$$|f(z)| \leq \frac{|z|^2}{2}$$

everywhere in Δ.

Problem 4: Show that all the zeroes of

$$p(z) := 3z^3 - 2z^2 + 2iz - 8$$

lie in the annulus $\{1 < |z| < 2\}$.

Problem 5: D is defined as the intersection of the sets $\{|z| < 1\}$ and $\{|z - \frac{1}{2}| > \frac{1}{2}\}$. Find a conformal map f of the region D onto the open unit disk Δ. Can you extend the mapping f to the closure of D? Is it conformal on part or all of the boundary?

Problem 6: A continuous mapping $f : X \to Y$ is called *proper* if, for any compact set $K \subset Y$, $f^{-1}(K)$ is compact in X. Prove that $f : \mathbb{C} \to \mathbb{C}$ is a polynomial if and only if it is a proper, entire function.

Problem 7: Let Δ be the open unit disk and B be the open unit ball in $\mathbb{C}^2$, i.e.

$$B := \{(z, w) : |z|^2 + |w|^2 < 1\} \subset \mathbb{C}^2.$$

Show that there cannot be any non-zero holomorphic function f, defined on Δ, whose graph is contained in B. (Hint: What happens when $|z| \to 1$?)