MATH313 – Preliminary Examination.

August 24, 2001

Instructions: Answer two out of the four questions. *You do not have to prove results which you rely upon, just state them clearly !*

Q1) Recall that, for an $n \times n$ real matrix A, the matrix norm induced by the 2–vector norm was found to be:

$$\|A\|_2 \; = \; \rho^{1/2}(A^T A),$$

where $\rho(\cdot)$ is the spectral radius of a matrix, namely,

$$\rho(B) \; = \; \max\{|\lambda| \mid \det(\lambda I - B) = 0\}.$$

Recall also that the condition number of an invertible matrix A with respect to the 2–norm is $\mathrm{Cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$. Suppose now that A and B are $n \times n$ real invertible matrices. Prove the following facts:

(i) $\mathrm{Cond}_2(A) \; \geq \; 1$.

(ii) $\mathrm{Cond}_2(A^T A) \; = \; (\mathrm{Cond}_2(A))^2$.

(iii) $\mathrm{Cond}_2(A) \; = \; \mathrm{Cond}_2(A^T)$.

(iv) $\mathrm{Cond}_2(AB) \; \leq \; \mathrm{Cond}_2(A)\mathrm{Cond}_2(B)$.

(v) $\mathrm{Cond}_2(\alpha A) \; = \; \mathrm{Cond}_2(A)$, where α is a nonzero scalar.

(vi) $\mathrm{Cond}_2(A) \; \geq \; |\lambda_1|/|\lambda_n|$, where $|\lambda_1| \geq \ldots \geq |\lambda_n| > 0$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Q2) a) Let $A^{(1)} = A \in \mathbb{R}^{n,n}$ be an invertible matrix which admits an LU–factorization without pivoting. Let $M_1, \ldots, M_{k-1}$ be elementary lower triangular matrices of order n and indices $1, \ldots, k-1$, respectively[1], for which $A^{(k)} = M_{k-1} \cdots M_1 A^{(1)}$ has zeros under its first $(k-1)$ diagonal entries. Partition A into the block partitioning

$$A = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right],$$

where $A_{1,1}$ is of size $(k-1) \times (k-1)$ and partition $A^{(k)}$ in conformity with A into

$$A^{(k)} = \left[\begin{array}{c|c} * & * \\ \hline 0 & A_k \end{array}\right].$$

Justify why $A_{1,1}$ is invertible and show that:

$$A_k = A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2}.$$

Finally, prove that if, in addition, A is symmetric, then A_k is symmetric.

b) Show that, in general (that is, not taking advantage of zero entries), the number of multiplication operations and division operations which are required to reduce an $n \times n$ matrix to an upper triangular matrix is

$$\frac{n^3}{3} - \frac{n}{3}.$$

(c) Show that if $A \in \mathbb{C}^{n,n}$ is an invertible matrix and $A = L_1 U_1 = L_2 U_2$ are LU–factorizations of A with the diagonal entries of L_1 and L_2 all 1's, then $L_1 = L_2$ and $U_1 = U_2$. [Carefully state all the results on which you rely, but do not prove these auxiliary result.]

Q3) (a) Prove: A qudrature formula $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ that uses the $n+1$ distinct nodes $x_0, \ldots, x_n$ and is exact of order at least n is interpolatory, that is,

$$\alpha_k = \int_a^b L_k(x)dx, \quad k = 0, \ldots, n,$$

[1] Recall that a matrix M is called an *elementary matrix of order* n *and index* i if M is an $n \times n$ matrix of the form $M = I - m_i e_i^T$, where $m_i = (\underbrace{0, \ldots, 0}_{i \text{ zeros}}, \underbrace{\mu_{i+1}, \ldots, \mu_n}_{\text{real numbers}})^T$ and e_i is the usual i–th coordinate vector in the n–dimensional space.

where

$$L_k(x) \;=\; \frac{\prod_{\substack{j=0\\ j\neq k}}^{n}(x-x_j)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(x_k-x_j)}, \quad k=0,\ldots,n.$$

(b) The Legendre polynomial of degree n is defined by

$$P_n(x) \;=\; \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\left(x^2-1\right)^n,$$

with $P_0(x) \equiv 1$. Calculate explicitly $P_1,\ldots,P_4$. Prove (verify) that for $k=0,1,\ldots,n-1$,

$$\int_{-1}^{1} x^k P_n(x)dx \;=\; 0.$$

(c) Use part (b) to conclude that $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$, when $m \neq n$, and that $\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1)$.

Q4) (a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n\cos^{-1}x), \quad n = 0,1,....$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1,1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm of a function on the interval $[-1,1]$.

(c) Obtain that $p(x) \approx 0.98516 + .11961x$ is the best approximation polynomial of order at most 1 to the function $f(x) = \sqrt{1+(1/4)x^2}$ over the interval $[0,1]$