303
August 2001

PhD Prelim Exam—Measure & Integration

For problems **1** and **2**, $(X,\mathcal{A},\mu)$ denotes an abstract measure space and all functions are real-valued and defined on X.

1. (a) Define what is meant by saying f is $\mathcal{A}$-measurable?

(b) Give precise definitions of the following modes of convergence:

(i) The sequence (f_n) converges to f μ*-almost everywhere* (abbrev. *AE*).

(ii) The sequence (f_n) converges to f μ*-almost uniformly* (abbrev. *AU*).

(iii) The sequence (f_n) converges to f *in measure* (abbrev. *M*).

(iv) The sequence (f_n) converges to f *in mean of order p* (abbrev. $\mathcal{L}^p$).

(c) Prove the two implications: $AU \to AE$ and $\mathcal{L}^p \to M$.

(d) Give a diagram for "modes of convergence" *in the general case* where a solid arrow means the first mode always implies the second mode, and where a broken arrow means convergence in the first mode implies the existence of a subsequence which converges in the second mode. Include all implications, even those that follow by transitivity (no additional proofs required).

2. (a) Give a precise statement of Fatou's Lemma. (no proof required).

(b) Suppose (f_n) is a sequence of nonnegative $\mathcal{A}$-measurable functions converging μ-a.e. to the function f, and suppose $\int f_n\,d\mu \to \int f\,d\mu < \infty$. Using only part (a) and properties of sequences of real numbers show that for every $A \in \mathcal{A}$ $\int_A f_n\,d\mu \to \int_A f\,d\mu$.

(c) Indicate where your proof depends on the assumption $\int f\,d\mu < \infty$, and give an example to show that this assumption cannot be omitted.

3. (a) Let $X = \{1,2,\ldots\}$, $\mathcal{A} = \mathcal{P}(X)$ (the power set of X), and let μ be counting measure (*i.e.*, the measure is ∞ if the set is infinite and equal to the cardinality otherwise). Define the function g on X by the rule $g(n) = n^{-1/p}$ where p is a fixed index in $[1,\infty)$. Show that $g \in \mathcal{L}^r$ iff $p < r \le \infty$. Hence deduce that $\mathcal{L}^r \not\subset \mathcal{L}^p$ for $p < r$.

(b) Let X and $\mathcal{A}$ be as in part (a) but let μ be the measure such that $\mu(\{n\}) = 1/n^2$ for all $n = 1,2,\ldots$. Define the function f on X by the rule $f(n) = n^{1/r}$ where r is a fixed index in $(1,\infty)$. Show that $f \in \mathcal{L}^p$ iff $1 \le p < r$. Hence deduce that $\mathcal{L}^p \not\subset \mathcal{L}^r$ for $p < r$.

(c) For a general X assume that $\mu(X) < \infty$ and $1 \le p < r < \infty$. Show that $\mathcal{L}^r \subset \mathcal{L}^p$ and, for all $f \in \mathcal{L}^r$, the following inequality holds

$$\|f\|_p \le \|f\|_r\,\mu(X)^{\frac{1}{p}-\frac{1}{r}}.$$

(Hint: Note that $|f|^p \in \mathcal{L}^{\frac{r}{p}}$ and $1 \in \mathcal{L}^s$ fo all $s \ge 1$.)

4. (a) Consider two measure spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ where $X = Y = [0,1]$, both σ-algebras are the Borel sets of $[0,1]$, μ is Lebesgue measure and ν is counting measure (see 3 (a) for the definition). If $D = \{(x,y) : x = y\}$, show that D belongs to the product σ-algebra, but that $\int \nu(D_x)\,d\mu(x) \ne \int \mu(D^y)\,d\nu(y)$. Why does this not contradict Tonelli's Theorem?

(b) Consider the reals with Lebesgue measure and the plane with the induced product measure. Let f be the function from the plane to the reals defined by $f(x,y) = 1$ if $x \ge 0$ and $x \le y < x+1$, $f(x,y) = -1$ if $x \ge 0$ and $x+1 \le y < x+2$, and $f(x,y) = 0$ otherwise. Show that $\int\left[\int f(x,y)\,dx\right]dy \ne \int\left[\int f(x,y)\,dy\right]dx$. State why this does not contradict Fubini's Theorem and verify any claim.

5. Let λ and μ be measures on the σ-algebra $\mathcal{A}$ for the space X. State what it means for λ to be *absolutely continuous* with respect to μ (symbolized $\lambda << \mu$). Define what is meant by a *Radon-Nikodym derivative* $d\lambda/d\mu$. Let λ and μ be σ-finite measures on $(X,\mathcal{A})$, let $\lambda << \mu$, and let $f = d\lambda/d\mu$. If g is a nonnegative $\mathcal{A}$-measurable function on X, show that $\int g\,d\lambda = \int g f\,d\mu$.