Preliminary Examination

Complex Analysis

2002

Instructions: Do all problems. Show your work in order to receive **ANY** credit. The terms region and domain mean the same thing. So do the terms complex analytic and holomorphic.

Problem 1: Suppose f is holomorphic in a region Ω that contains the closed unit disk and $|f(z)| < 1$ when $|z| = 1$. How many fixed points (solutions to $z = f(z)$) must f have in the open unit disk Δ.

Problem 2: Suppose f is an entire function and there are constants A and B and a positive integer k so that

$$|f(z)| \leq A + B|z|^k$$

for all z. Prove that f must be a polynomial.

Problem 3: Compute (justifying your computations)

(*i*) $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx.$

(*ii*) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ where $a > b > 0$.

Problem 4: Suppose f is holomorphic and non-zero in the simply connected domain Ω.

(i) If n is any positive integer, prove that there exists a function g, holomorphic in Ω and satisfying $g^n = f$.

(ii) How many holomorphic solutions does $g^3 = f$ have in a small disk about 0 if $f(z) := z^4 + 16$.

(iii) Find the Taylor polynomial of degree 5 for the holomorphic solution g in part (ii) for which $g(0) \in \mathbb{R}$.

Problem 5: Suppose D is a region in $\mathbb{C}$ and $H(D)$ denotes the space of functions which are holomorphic in D. Let (f_n) be a locally bounded sequence in $H(D)$ and $f \in H(D)$. Assume

$$A := \{z \in D \mid \lim f_n(z) = f(z)\}$$

has a limit point in D. Show that there exists a subsequence of (f_n) which converges to f uniformly on compact subsets of D.

Problem 6: In a domain D containing 0, a function

$$\begin{aligned} f \ :\ & D && \to && \mathbb{C} \\ :\ & (x,y) && \mapsto && f(x,y) = u(x,y) + iv(x,y) \end{aligned}$$

is complex harmonic if both u and v are (real) harmonic in D. You may assume that f admits an absolutely convergent double power series expansion

$$f(z,\bar{z}) = \sum_{n,m=0}^{\infty} a_{nm} z^n \bar{z}^m$$

and that the usual differentiation and integration rules for power series in one variable are valid here.

(i) Under what conditions on the coefficients a_{nm} is f holomprphic in D?

(ii) Under what conditions on the coefficients a_{nm} is f complex harmonic in D?