MATH 313 August 2002, PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Q1) **(a)** Suppose that $p(x)$ is a polynomial of degree at most n which has $n+1$ distinct roots. Show that $p(x) \equiv 0$. Use this result to show that the polynomial p_n, of order at most n, which interpolates a function f at $n+1$ distinct points $x_0, \ldots, x_n$ is unique. [Assume that the values which f takes at these points are $f_0, \ldots f_n$, respectively.]

(b) Suppose that $f \in C^{n+1}[a,b]$ and that $x_0, \ldots, x_n$ are $n+1$ distinct points in the interval. Let p_n be the interpolation polynomial for f on $x_0, \ldots, x_n$. Let $e_n(x) = f(x) - p_n(x)$ denote the error function on $[a,b]$. Show that for each point $x \in [a,b]$, there is a point $\xi_x \in (a,b)$ such that

$$e_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

(c) A function f is defined on the interval $[0,1]$ and its derivatives satisfy that $|f^m(x)| \le m!$, for all $x \in [0,1]$ and for all $m = 0, 1, 2, \ldots$. For any $0 < q < 1$, let $p_n(x)$, $n \ge 0$, be the interpolation polynomial of degree at most n which interpolates f at $x_0 = 1, x_1 = q, x_2 = q^2, \ldots, x_n = q^n$. Show that

$$\lim_{n\to\infty} p_n(0) = f(0).$$

Taking $q = 1/2$ and $n = 10$, find an upper estimate on $|p_{10}(0) - f(0)|$.

Q2) The following compactness theorem is known: *Let V be a finite dimensional normed vector space and W be a closed subset of V. If there exists a constant $M > 0$ such that $\|w\| \le M$ for all $w \in W$, then any sequence in W has a convergent subsequence.*

Define P_n to be the vector space of polynomials of degree at most n and $\|f\| = \max_{0\le x\le 1} |f(x)|$ for any continous function $f \in C[0,1]$.

(a) Show that for any $f \in C[0,1]$, there exists a polynomial $p^* \in P_n$ which minimizies the uniform norm of $\|f - q\|$ for any $q \in P_n$.
(Hint: let $\inf_{w\in W} \|w - f\| = \alpha$. Then there exists a sequence $\{w_i\} \subset W$ such that $\|w_i - f\| \to \alpha$ as $i \to \infty$. The sequence $\{w_i\}$ is called a minimizing sequence.)

(b) Define a set on rational functions

$$R_{n,m} = \{\tfrac{p(x)}{q(x)} : \quad p \in P_n \text{ and } q \in P_m\ , \|q\| = 1,\ q > 0 \text{ on } [0,1],$$

$$p \text{ and } q \text{ have no common factors.}\}\ .$$

Our Goal: Given $f \in C[0,1]$, prove the existence of $r^* \in R_{n,m}$ such that it minimizes the uniform norm of $\|f - r\|$ for any $r \in R_{n,m}$.

Let p_i/q_i be a minimizing sequence. Show that there exists a constant M such that $\|q_i\|$, $\|p_i/q_i\|$ and $\|p_i\|$ are all bounded by M for all i.

(c) By Q2a, we can assume that (a subsequence of) p_i and q_i converge to $p \in P_n$ and $q \in P_m$, respectively. Explain why $q \geq 0$ and can have at most finite number of roots of even multiplicity in $[0,1]$.

(d) Let z be a root of q, explain why z has to be a root of p of at least the same multiplicity. (Hint: $\|p_i/q_i\| \leq M$ from part Q2b). Hence try to finish the proof for our goal stated in Q2b.

Q3) (a) Recall that the 1–norm of a vector $x = (x_1, \ldots, x_n) \in C^n$ is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$. Show that for $n \times n$ matrix $A = (a_{i,j}) \in C^{n,n}$, the 1–matrix norm induced by the 1–vector norm, that is, by

$$\|A\|_1 \;=\; \max_{\|x\|_1=1, x\in C^n} \|Ax\|_1,$$

is given by

$$\|A\|_1 \;=\; \max_{1\leq j\leq n} \sum_{i=1}^n |a_{i,j}|.$$

(b) Recall that for a matrix $B = (b_{i,j}) \in C^{n,n}$, $\|B\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |b_{i,j}|$ and that if B is invertible, then $\text{cond}_\infty(B) := \|B\|_\infty \|B^{-1}\|_\infty$.

Suppose now that $A = (a_{i,j}) \in C^{n,n}$ is an invertible matrix with $\sum_{j=1}^n |a_{i,j}| = 1$, $1 \leq i \leq n$. Show, first, that if D is any invertible diagonal matrix, then $\|DA\|_\infty = \|D\|_\infty$ and use this to show that

$$\text{cond}_\infty(A) \;\leq\; \text{cond}_\infty(DA),$$

Discuss the follwoing problem: Can the numerical stability of solving the system $Ax = b$, where A is as above, be improved by scaling the rows of the matrix A and the vector b by a diagonal matrix D, namely, by solving instead the system $A'x = b'$, where $A' = DA$ and $b' = Db$, for some invertible diagonal matrix D.

Q4) (a) Consider the uniform partition of the interval $[0, 2\pi]$,

$$x_k = \frac{2pik}{N}, \quad k = 0, ..., N-1, \quad N = 2M + 1.$$

Show that there exists a unique trigonometric polynomial

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M} (A_h \cos(hx) + B_h \sin(hx))$$

such that

$$\Psi(x_k) = y_k, \quad y_k \in C, \quad k = 0, ..., N-1.$$

(b) Show that if $y_k, k = 0, ..., N-1$ are real numbers, then A_h and B_h are also real numbers.