Ph.D. Exam for **Math 303**
Summer, 2002

1. Prove that $\lim_{a\to\infty}\int_1^a \frac{1}{x}\sin x\,dx$ exists but that the function $\frac{1}{x}\sin x$ is not Lebesgue integrable on $[1,\infty)$.

2. Show that if F is a non-decreasing function on $[a,b]$, then $F(b)-F(a)\geq \int_a^b F'(t)\,dt$. Give a meaning to the difference between the two quantities when F is right continuous by relating them to the Lebesgue decomposition of F (or of the associated measure), and an example showing that equality does not always hold.

3. Prove that the space $L_p(X,\mathcal{M},\mu)$, is complete for $1\leq p<\infty$.

4. Assume the Borel set $A\subset[0,1]$ satisfies the following property: there exists $0\leq\tau<1$ such that $m(A\cap I)\leq\tau m(I)$ for all intervals $I\subset[0,1]$. Prove that $m(A)=0$. (Here m is any finite Borel measure on [0,1].)

5. Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\boldsymbol{\mu})$ be σ-finite measure spaces, and let $f:S\times T\mapsto\mathbf{R}$ be a $\mathcal{S}\otimes\mathcal{T}$-measurable function. Let $p\geq 1$. Show that if $f(x,y)$ is in $L_p(X,\mathcal{S},\mu)$ for every fixed $y\in Y$, then the integral $\int f(x,y)d\nu(y)$ is also in $L_p(X,\mathcal{S},\mu)$, even better, prove the generalized Minkowsky's inequality:

$$\left(\int_X\left[\int_Y |f(x,y)|d\nu(y)\right]^p d\mu(x)\right)^{1/p}\leq\int_Y\left(\int_X |f(x,y)|d\mu(x)\right)^{1/p} d\nu(y).$$