M340 — Preliminary Exam in Complex Analysis

1. (a) Show that there is a complex differentiable function defined on the set $\Omega = \{z \in \mathbb{C} : |z| > 4\}$, whose derivative is $\dfrac{z}{(z-1)(z-2)(z-3)}$?

   (b) Is there a complex differentiable function on Ω whose derivative is $\dfrac{z^2}{(z-1)(z-2)(z-3)}$?

2. Evaluate $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\, d\theta$.

3. Suppose f and g are entire functions and $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$. Prove that there is a constant c such that $f = cg$.

4. (a) Prove that every one-to-one conformal mapping of $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ onto itself is a linear fractional (Möbius) transformation.

   (b) Prove that every one-to-one conformal mapping of $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ onto a disc $B = \{az \in \mathbb{C} : |z - a| \le r\}$ for some $a \in \mathbb{C}$ and $r > 0$ is a linear fractional transformation.

5. Show that $f(z) = \dfrac{z}{e^z - 1}$ has a removable singularity at $z = 0$ and that f has a power series expansion $f(z) = \sum_{n=1}^{\infty} c_n z^n$. Calculate c_0 and c_1 and show that $c_{2n+1} = 0$ for $n \ge 1$, i.e. $\sum_{n=1}^{\infty} c_n z^n$ is an even function). Find the radius of convergence of the series.

6. Let $0 < r < R$ and $A = \{z \in \mathbb{C} : r \le |z| \le R\}$. Show that there exists a positive number $\epsilon > 0$ such that for each polynomial p

$$\sup_{z \in A} \left|p(z) - \frac{1}{z}\right| \ge \epsilon$$

7. Evaluate the following real integral by using residues:

$$\int_0^{\infty} \frac{\cos x}{1 + x^2}\, dx = \frac{1}{2} \int_0^{\infty} \left(\frac{e^{ix}}{1 + x^2} + \frac{e^{-ix}}{1 + x^2}\right) dx$$