MATH 313 August 2003, PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Q1) **(a)** Recall that the 1–norm of a vector $x = (x_1, \ldots, x_n) \in C^n$ is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$. Show that for $n \times n$ matrix $A = (a_{i,j}) \in C^{n,n}$, the 1–matrix norm induced by the 1–vector norm, that is, by

$$\|A\|_1 \;=\; \max_{\|x\|_1=1, x\in C^n} \|Ax\|_1,$$

is given by

$$\|A\|_1 \;=\; \max_{1\le j\le n} \sum_{i=1}^n |a_{i,j}|.$$

(b) Recall that for a matrix $B = (b_{i,j}) \in C^{n,n}$, $\|B\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |b_{i,j}|$.

Suppose now that $A = (a_{i,j}) \in C^{n,n}$ is an invertible matrix with $\sum_{j=1}^n |a_{i,j}| = 1$, $1 \le i \le n$. Show, first, that if D is an invertible diagonal matrix, then $\|DA\|_\infty = \|D\|_\infty$ and use this to show that

$$\text{cond}_\infty(A) \;\le\; \text{cond}_\infty(DA).$$

Where, for a nonsingular matrix B, $\text{cond}_\infty(B) = \|B\|_\infty \|B^{-1}\|_\infty$.

(c) Discuss the following problem: Can the numerical stability of solving the system $Ax = b$, where A is as above, be improved by scaling the rows of the matrix A and the vector b by a diagonal matrix D, namely, by solving instead the system $A'x = b'$, where $A' = DA$ and $b' = Db$, for some invertible diagonal matrix D?

Q2) **(a)** Determine the polynomial of degree at most $n-1$ which best approximates the polynomial

$$Q(x) \;=\; a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

on the interval $[a, b]$ and show that its maximum deviation from Q is given by

$$\frac{1}{2^{n-1}} \left(\frac{b-a}{2}\right)^n a_0.$$

(b) Show that the polynomial of degree at most 2 which best approximates the polynomial $ax^3 + bx^2 + cx + d$ on the interval $[-1, 1]$ is given by

$$bx^2 + \left(c + \frac{3a}{4}\right) x + d.$$

(Recall that Chebyshev polynomials satisfy the three term recursion, $T_0 = 1$, $T_1 = x$, $T_{n+1} = 2xT_n - T_{n-1}$).

Q3) Let $w(x)$ be a positive continuous function on $[a, b]$. For $j = 1, 2, ...,$ let $p_j(x)$ be the corresponding monic orthogonal polynomial of degree j, i.e.,

$$p_j(x) = x^j + a_1 x^{j-1} + \cdots + a_j,$$

such that $(p_j, p_k) = \int_a^b w(x)p_j(x)p_k(x)dx = 0$ if $j \neq k$. In particular $p_0(x) = 1$.

(a) Prove that the roots $x_1, .., x_n$ of $p_n(x)$ are real, simple and lie in (a, b).

(b) Prove that $p_n(x)$ satisfy a three term recurrence relation, i.e.,

$$p_{i+1}(x) = (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x), \quad \imath \geq 0,$$

where $p_{i-1} = 0, \quad \gamma_1 = 0$, and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad i \geq 0, \quad \gamma_{i+1}^2 = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad i \geq 1.$$

(c) For $a = -1; b = 1; w(x) = 1$; find $p_1(x)$ and $p_2(x)$.

Q4) **(a)** Consider the uniform partition of the interval $[0, 2\pi]$,

$$x_k = \frac{2\pi k}{N}, \quad k = 0, ..., N - 1, \quad N = 2M + 1.$$

Show that there exists a unique trigonometric polynomial

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M}(A_h cos(hx) + B_h sin(hx))$$

such that

$$\Psi(x_k) = y_k, \quad y_k \in C, \quad k = 0, ..., N - 1.$$

(b) Show that if $y_k, k = 0, ..., N-1$, are real numbers, then A_h and B_h are also real numbers.