Summer 03
Dept of Math
Univ of CT

PhD Prelim exam — Measure & Integration (Math 303)

1. Let $(X, \mathscr{F}, \mu)$ be a measure space. Let f be a $\mathbb{R}$–valued measurable function defined on X, and for $p \in (0,\infty)$, define

$$\varphi(p) = \int_X |f|^p \, d\mu := \|f\|_{L^p}^p.$$

Let

$$E = \{p: \varphi(p) < \infty\},$$

and assume $\|f\|_{L^\infty} < \infty$.

(a) Prove that either $E = \emptyset$, or E is an unbounded subinterval of $(0,\infty)$, and that if $\varphi(p) > 0$ for some (equivalently, all) $p > 0$ and $E \neq \emptyset$, then $\log \varphi$ is convex on E.

(b) Prove: if $E \neq \emptyset$, then φ is continuous on E.

(c) Is E necessarily open? Closed? (Affirmative replies must be proved; otherwise, if reply is negative, give couterexample.)

2. Prove: for $p \in [1,\infty)$, if $f \in L^p(\mathbb{R},m)$, and $g \in L^1(\mathbb{R},m)$, then $\|f*g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.
(m denotes Lebesgue measure on $\mathbb{R}$, and

$$f*g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Note: every application of a major theorem must be clearly cited.)

3. Let $(X, \mathscr{F}, \mu)$ be a finite measure space.

(a) Prove or disprove: if a sequence (f_n) of $\mathbb{R}$–valued $\mathscr{F}$–measurable functions on X converges a.e. (μ) on X, then (f_n) converges in measure (μ).

(b) Prove or disprove: if a sequence (f_n) of $\mathbb{R}$–valued $\mathscr{F}$–measurable functions on X converges in measure (μ) on X, then (f_n) converges a.e. (μ).

(c) Prove or disprove: if a sequence (f_n) of $\mathbb{R}$–valued $\mathscr{F}$–measurable functions on X is Cauchy in $L^1(\mu)$, then (f_n) converges in measure (μ).

4. Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) Prove: if $f \in L^1(\mu)$, then for every $\epsilon > 0$ there exist $\delta > 0$ such that if $\mu(A) < \delta$ $(A \in \mathcal{F})$, then

$$\int_A |f| \, d\mu < \epsilon.$$

(b) A sequence (f_n) in $L^1(\mu)$ is said have *uniformly absolutely continuous integrals* if for every $\epsilon > 0$ there exist $\delta > 0$ such that $\mu(A) < \delta$ $(A \in \mathcal{F})$ implies

$$\int_A |f_n| \, d\mu < \epsilon \quad \text{for all } n = 1, \ldots .$$

Suppose $\mu(X) < \infty$. Prove: if (f_n) in $L^1(\mu)$ has *uniformly absolutely continuous integrals*, and $f_n \to f$ a.e. (μ), then $f_n \to f$ in $L^1(\mu)$.

(c) Show that the result in part (b) implies the *Lebesgue Dominated Convergence Theorem.*

5. Let μ be a signed measure on the σ-algebra $\mathcal{F}$ in X. Prove (by applying the Radon–Nikodym theorem) that there exists a unique $\mathcal{F}$-measurable function h on X, such that $|h(x)| = 1$ a.e. (μ), and

$$\mu(A) = \int_A h \, d|\mu| \quad \text{for all } A \in \mathcal{F},$$

where $|\mu|$ is the total variation measure.