University of Connecticut
Department of Mathematics
Preliminary Exam – Risk Theory Section (Math 395)
August 19, 2003, 9AM

There are 5 questions, all of which should be answered. Show your calculations and give the reasons that justify your steps, although you do not need formally to prove the results. You may refer to a summary of key formulas for a variety of distributions that is attached to this examination. You may use any hand-held calculator. There are 3 hours for the exam.

1. A surplus process is defined by $u(t) = u + ct - S(t)$ where $S(t)$ is a compound Poisson process with $\lambda = 600$ and individual claim distribution $p(x) = 3e^{-3x}$. The premium accumulation rate is $c = 200$. Write an exact formula (with no unknown coefficients) for the probability of ruin $\psi(u)$ as a function of the initial surplus u.

2. Answer the same question if the premium accumulation rate is $c = 210$.

3. Individual loss amounts (ground up) this year follow a Weibull distribution with mean 250 and standard deviation 250. Next year you confidently expect loss amounts to inflate by 10% uniformly across the board. What will be the standard deviation next year for loss amounts that are subject to 100 deductible per loss with losses after deductible limited to 1,000 per loss? Please answer for the "per loss" variable, not the "per payment" variable.

4. A certain claim frequency variable is known to follow a Poisson distribution for any given insured person, but the Poisson frequency λ varies among the insured population of 300,000 people. In fact, λ is distributed across the population as the sum of 10 identically and independently distributed exponential variables, each with mean 600. Suppose the insured population increases to 500,000 people, but in a manner such that all risk characteristics per person stay the same. What are the mean, variance and third central moment of the frequency of claims from the new population of 500,000 people.

5. Let $S(t) = X_1 + \ldots + X_{N(t)}$ where $N(t)$ is Poisson with frequency 10t and the X's are independent and identically distributed with the property that the conditional distribution of $(S(t) \mid N(t) = N^*)$ is a gamma distribution with parameter $\alpha = N^*$ and mean $2N^*$ for any integer N^*. Let $L = \max_{t\geq 0} \{(S(t) - 22t)_+\}$ be the maximum aggregate loss random variable with premium rate $c = 22$. Express L as $L=K_1+\ldots+K_M$ where M is a random counting variable and the K's are independent and identically distributed. Approximate K (by rounding) using a discrete distribution with whole integer units. Calculate the resulting approximate values for (a) the probability $\psi(4)$ of ruin from a starting surplus of 4 and (b) the expected value $E[(L-4)_+ \mid L>4]$ of the largest excess of accumulated claims over accumulated premium plus starting surplus of 4, contingent upon ruin occurring.

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0,\ x > 0$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0.$$

When $\alpha \leq 0$ the integral does not exist. In that case, define

$$\Gamma(\alpha) G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \qquad x > 0.$$

Integration by parts produces the relationship

$$\Gamma(\alpha) G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{\Gamma(\alpha+1)}{\alpha} G(\alpha+1; x)$$

which allows for recursive calculation because for $\alpha > 0$, $\Gamma(\alpha)G(\alpha;x) = \Gamma(\alpha)[1 - \Gamma(\alpha; x)]$.

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \qquad a > 0,\ b > 0,\ 0 < x < 1.$$

A.2 Transformed beta family

A.2.3 Two-parameter distributions

A.2.3.1 Pareto—α, θ

$$\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
E[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
E[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
E[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
E[X \wedge x] &= -\theta \log\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
E[(X \wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}$$

A.2.3.2 Inverse Pareto—τ, θ

$$\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
E[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
E[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
E[(X \wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}$$

A.2.3.3 Loglogistic—γ, θ

$$\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
E[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
E[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}$$

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$
\begin{aligned}
f(x) &= \frac{\alpha^2 (x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
E[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha - k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
\text{mode} &= \theta \left(\frac{\alpha - 1}{\alpha^2 + 1}\right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
E[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1 - k/\tau; u) + x^k [1 - u^\tau], \quad k > -\tau^2 \\
\text{mode} &= \theta (\tau - 1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\
M(t) &= (1 - \theta t)^{-\alpha} \qquad E[X^k] = \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
E[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha, \quad \text{if } k \text{ is an integer}
\end{aligned}
$$

$$
\begin{aligned}
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
&= \alpha(\alpha+1)\cdots(\alpha+k-1)\theta^k \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
\text{mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.3.2.2 Inverse gamma—α, θ

$$\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)} \qquad F(x) = 1 - \Gamma(\alpha; \theta/x) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha \qquad E[X^k] = \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
&= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \text{ all } k \\
&= \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \ k \text{ is an integer} \\
\text{mode} &= \theta/(\alpha + 1)
\end{aligned}$$

A.3.2.3 Weibull—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau (x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} \qquad F(x) = 1 - e^{-(x/\theta)^\tau} \\
E[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau \\
\text{mode} &= \theta \left(\frac{\tau - 1}{\tau} \right)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}$$

A.3.2.4 Inverse Weibull—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau (\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} \qquad F(x) = e^{-(\theta/x)^\tau} \\
E[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau)\{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \quad \text{all } k \\
&= \theta^k \Gamma(1 - k/\tau) G[1 - k/\tau; (\theta/x)^\tau] + x^k \left[1 - e^{-(\theta/x)^\tau}\right] \\
\text{mode} &= \theta \left(\frac{\tau}{\tau + 1} \right)^{1/\tau}
\end{aligned}$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad E[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
E[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
E[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
E[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}
$$

A.3.3.2 Inverse exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
E[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1-k) G(1-k; \theta/x) + x^k (1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}
$$

A.4 Other distributions

A.4.1.1 Lognormal—μ,σ (μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\log x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
E[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
E[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}
$$

A.4.1.2 Inverse Gaussian—μ, θ

$$
\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\theta z^2}{2x}\right\}, \quad z = \frac{x-\mu}{\mu} \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu} \\
M(t) &= \exp\left[\frac{\theta}{\mu}\left(1 - \sqrt{1 - \frac{2t\mu^2}{\theta}}\right)\right] \qquad E[X] = \mu, \quad Var[X] = \mu^3/\theta \\
E[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right]
\end{aligned}
$$

A.4.1.3 Single-parameter Pareto—α, θ

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta \qquad F(x) = 1 - (\theta/x)^\alpha, \quad x > \theta$$

$$E[X^k] = \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha \qquad E[(X \wedge x)^k] = \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}$$

$$\text{mode} = \theta$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.5 Distributions with finite support

For these two distributions, the scale parameter θ is assumed known.

A.5.1.1 Generalized beta—a, b, θ, τ

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^\tau$$

$$F(x) = \beta(a, b; u)$$

$$E[X^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau$$

$$E[(X \wedge x)^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a+k/\tau, b; u) + x^k[1 - \beta(a, b; u)]$$

A.5.1.2 beta—a, b, θ

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta$$

$$F(x) = \beta(a, b; u)$$

$$E[X^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a$$

$$E[X^k] = \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)}, \quad \text{if } k \text{ is an integer}$$

$$E[(X \wedge x)^k] = \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) + x^k[1 - \beta(a, b; u)]$$

Appendix B

An Inventory of Discrete Distributions

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ E[N] &= \lambda, \quad Var[N] = \lambda \qquad P(z) = e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= 1/(1+\beta), \quad a = \beta/(1+\beta), \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ E[N] &= \beta, \quad Var[N] = \beta(1+\beta) \qquad P(z) = [1 - \beta(z-1)]^{-1} \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -q/(1-q), \quad b = (m+1)q/(1-q) \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\ E[N] &= mq, \quad Var[N] = mq(1-q) \qquad P(z) = [1 + q(z-1)]^m \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \beta/(1+\beta), \quad b = (r-1)\beta/(1+\beta) \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ E[N] &= r\beta, \quad Var[N] = r\beta(1+\beta) \qquad P(z) = [1 - \beta(z-1)]^{-r} \end{aligned}$$