Department of Mathematics
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PhD Prelim Exam— Math 303

Notation: Throughout the exam, m denotes Lebesgue measure on $\mathbb{R}$, and $(X,\mathscr{F},\mu)$ denotes an abstract σ–finite measure space. All functions are scalar–valued, and (unless otherwise stated) the scalar field is the field of real numbers $\mathbb{R}$.

Note: Every application of a major theorem must be clearly cited. All major results used and cited are collected in the first problem below.

10 pts 1. State clearly and completely all major results that you use and cite in your work.

2. True or false? (If true, then prove it; if false, then give a counterexample. No credit will be given for guesses.)

(a) If a sequence of Lebesgue–measurable functions on $[0,1]$ converges a.e. (m), then it converges in measure (m).

(b) If a sequence of Lebesuge–measurable functions (f_n) converges to f in $L^1([0,1],m)$, then $f_n \to f$ a.e. (m) on $[0,1]$.

30 pts

(c) If $f \in L^1(X,\mathscr{F},\mu)$, $f_n \in L^1(X,\mathscr{F},\mu)$ for all $n \in \mathbb{N}$, $f_n \to f$ a.e. (μ), and $\int_X |f_n|\, dm \longrightarrow \int_X |f_n|\, dm$, then $f_n \to f$ in $L^1(X,\mathscr{F},\mu)$.

(d) If $f \in L^1(\mathbb{R},m)$, and f is uniformly continuous on $\mathbb{R}$, then $f \in C_0(\mathbb{R})$.

(e) If $f \in L^1(X,\mathscr{F},\mu)$ and $\mathfrak{A} \subset \mathscr{F}$ is a σ–algebra, then there exists a $\mathfrak{A}$–measurable function g on X, such that for every $A \in \mathfrak{A}$,

$$\int_A g\, d\mu = \int_A f\, d\mu .$$

3. (a) Suppose f is a non–negative function in $L^1(X,\mathscr{F},\mu)$. Prove that

$$(*) \qquad \lim_{\lambda\to\infty} \lambda\, \mu(f \geq \lambda) = 0.$$

20 pts

(b) Produce a non–negative Lebesgue–measurable function f on $[0,1]$ such that $(*)$ above holds with $\mu = m$, but $f \notin L^1([0,1],m)$. (A small hint: consider a monotone function...)

(c) Suppose μ is finite, and that f is a non-negative $\mathscr{F}$-measurable function on X. Prove that $f \in L^1(X,\mathscr{F},\mu)$ if and only if

$$\sum_{n=1}^{\infty} \mu(f > n) < \infty.$$

(Hint: write $1 = n - (n-1)$, and sum by parts.)

20 pts

4. For $p \in [1,\infty)$, prove that if $f \in L^p(\mathbb{R},m)$ and $g \in L^1(\mathbb{R},m)$, then

$$f*g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy$$

is well-defined for almost all x (m), and that $f*g \in L^p(\mathbb{R},m)$ with

$$\|f*g\|_{L^p} \le \|f\|_{L^p} \|g\|_{L^1}$$

20 pts

5. Let $\mathfrak{M}$ denote the σ-algebra of Lebesgue-measurable subsets of $(0,1)$. Consider a $\mathbb{R}$-valued $\sigma(\mathfrak{M}\times\mathscr{F})$-measurable function f on $(0,1)\times X$ that satisfies

(i) for each $t \in (0,1)$, $f(t,\cdot) \in L^1(\mu)$;

(ii) for each $x \in X$, $f(\cdot,x)$ is differentiable on $(0,1)$;

(iii) there exists $g \in L^1(\mu)$ such that for all $s \in (0,1)$ and $x \in X$,

$$\left|\frac{\partial f}{\partial t}(s,x)\right| \le g(x).$$

Define $\varphi : (0,1) \to \mathbb{R}$ by

$$\varphi(t) = \int_X f(t,x)\, \mu(dx).$$

(a) Prove that φ is differentiable at every point of $(0,1)$, and that for all $s \in (0,1)$,

$$\varphi'(s) = \int_X \frac{\partial f}{\partial t}(s,x)\, \mu(dx).$$

(b) Determine whether φ is absolutely continuous on $(0,1)$.