Algebra Prelim Exam August 2005

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1a) Show that a group of order 45 is abelian.

1b) Is every group of order p^2q, with p and q distinct primes, abelian?

2) Suppose G is a group. Recall that G satisfies the ascending chain condition on subgroups, if for any subgroups $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ there is a positive integer i such that $H_i = H_{i+1} = H_{i+2} = \cdots$. Also G satisfies the descending chain condition on subgroups if for any subgroups $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ there is an integer i such that $H_i = H_{i+1} = H_{i+2} = \cdots$.

(a) Show that every finitely generated abelian group satisfies the ascending chain condition.

(b) Which finitely generated abelian groups satisfy the descending chain condition?

3) Let $R = Z[x, y]$.

(a) If I is a principal ideal of R, show that there are only finitely many principal ideals of R which contain I.

(b) Show that (x, y) is a prime ideal of R which is not maximal.

4) Let V be a vector space over the field F and B_1 and B_2 two bases for V. Show that if B_1 has infinite cardinality then B_2 also has infinite cardinality.
(You may not just quote the uniqueness of dimension for a vector space.)

5) Let n be a positive integer, $GL(n, C)$ the group of invertible $n \times n$ complex matrices, and G a finite group. Suppose $\Psi : G \longrightarrow GL(n, C)$ is a group homomorphism. If $g \in G$, show that $\Psi(g)$ is a diagonalizable matrix.

6) Suppose R is a commutative ring with 1, M an R-module and

$$\Psi : M \longrightarrow R$$

an onto R-module homomorphism. Show that

$$M = Ker(\Psi) \oplus B$$

for some submodule $B \subseteq M$ with $B \simeq R$.