MATH 310, Preliminary Exam

DEPARTMENT OF MATHEMATICS
University of Connecticut

August, 2005

NAME: ______________________________ SIGNATURE: ______________________________

DO SIX OF THE SEVEN QUESTIONS!

1. a) What is the definition of a compact linear operator from a Banach space X to itself;

b) Give an example of an operator for $X = L^p([0,1])$ which is a compact linear operator and explain why;

c) Give an example of an operator for $X = L^p([0,1])$ which is NOT a compact linear operator and explain why;

2. a) What is the definition of weak convergence of a sequence $\{x_n\}$ in a Hilbert space H;

b) Prove that a strongly convergent sequence is also a weakly convergent sequence in H;

c) Give an example of a weakly convergent sequence which is NOT strongly convergent in l^2 and explain;

3. a) Give an example of a distribution which can NOT be identified with a continuous function in R and explain why.

b) Define $\delta(0)$ as a distribution;

c) If $T(\phi) = \phi(0) + \phi(1)$ for every $\phi \in \mathcal{D}(\mathcal{R})$, find ∂T the derivative of T.

4. a) Suppose f is an operator from Banach space X to itself. Give the definition of f being Fréchet differentiable at a point $x \in X$.

b) Let $X = C[0,1]$ with sup-norm. Let $t_i \in [0,1]$ and $v_i \in C[0,1]$, and define $f(x) = \Sigma_{i=1}^{n}(x(t_i))^2 v_i$. Prove that f is Fréchet differentiable at all points of X and give a formula for f'.

5. Find a function in $C^1[0,1]$ that minimizes the integral $\int_0^1 [(u'(t))^2 + u(t)]dt$ with constraints $u(0) = 0$ and $u(1) = 1$.

6. Find an orthonormal basis for $L^2[0,1]$ by considering the Sturm-Liouville operator $Ax = x'' + x$ with $x(0) = x'(1) = 0$. Explain the reasons (theory) behind your method.

7. Let $[u_n]$ be an orthonormal sequence in a Hilbert space and let $[\lambda_n]$ be a bounded sequence in R. Prove that the operator $Ax = \Sigma\lambda_n < x, u_n > u_n$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.