Math 340
Preliminary Exam in Complex Analysis
August 2005

1. (a) Prove the Minimum Principle for harmonic functions, i.e., show that if u is harmonic in a region Ω and u attains a minimum at a point $z_0 \in \Omega$, then u is constant.

   (b) Suppose f is analytic in the unit disk Δ and continuous in $\overline{\Delta}$, and $f(z)$ is real for $|z| = 1$. Show that f is a constant.

2. Let f and g be analytic and non-zero in a connected open set Ω. Suppose also that there exists a sequence of complex numbers $z_n \in \Omega$ so that $z_n \to p \in \Omega$ and for all positive integers n,
$$\frac{f'(z_n)}{f(z_n)} = \frac{g'(z_n)}{g(z_n)}.$$
Show that there is a constant c so that $g = cf$.

3. Show that if f is analytic in Δ and $|f(z)| \leq 1$ in Δ, then for all $z \in \Delta$,
$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

4. Suppose f is an entire function, f is bounded for $\operatorname{Re} z \geq 0$, and f' is bounded for $\operatorname{Re} z \leq 0$. Prove that f is a constant.

5. (a) Suppose $R(z)$ is a rational function with no poles on the unit circle. Prove that
$$\int_{|z|=1} R(z)\, dz = \int_{|w|=1} R(1/w)\, \frac{dw}{w^2}.$$

   (b) Use Part (a) to evaluate the integral
$$\int_{|z|=1} \frac{z^{11}}{12z^{12} - 4z^9 + 2z^6 - 4z^3 + 1}\, dz.$$

6. Suppose (g_n) is a sequence of non-constant entire functions. If the functions g_n converge uniformly to a function g, what can you conclude about the sequence? When is the limit entire?

   In the following parts, assume $g_n, n \in I\!N$ is a sequence of entire functions, having only real zeros. Further, suppose that the functions g_n converge uniformly on compact subsets of $\mathbb{C}$ to an entire function g.

   (a) Prove that if g is not identically zero, then g has only real zeros.

   (b) Is it possible for g to be identically zero? Explain your answer.