Department of Mathematics, University of Connecticut August 2005

Ph.D. Preliminary Examination
Math 303 – Measure and Integration

General instructions. Do any four of the five problems.

1. Let $(X, \mathcal{A})$ be a measurable space, that is, X is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of X. Let $< \mu_k >_{k=1}^{\infty}$ be a sequence of (positive) measures on $\mathcal{A}$ such that $\mu_k(X) = 1$ for all k, and let μ be a measure on $\mathcal{A}$ such that, for each $A \in \mathcal{A}$, $\lim_{k\to\infty} \mu_k(A) = \mu(A)$ (so necessarily μ is a positive measure and $\mu(X) = 1$).

   (a) Show that if $f : X \to \mathbb{R}$ is bounded and $\mathcal{A}$-measurable, then
   $$\int f\, d\mu = \lim_{k\to\infty} \int f\, d\mu_k.$$
   [*Hint:* It's *simple.*]

   (b) Show that if $f : X \to \mathbb{R}$ is nonnegative-valued and $\mathcal{A}$-measurable, then
   $$\int f\, d\mu \le \liminf_{k\to\infty} \int f\, d\mu_k.$$

   (c) Now we specialize: let $X = \mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$, let $\mathcal{A}$ consist of all subsets of X, let $f : X \to \mathbb{R}$ be $f(n) = n$, and let the measures μ_k and μ be determined by
   $$\mu_k(\{0\}) = 1 - k^{-1}, \quad \mu_k(\{k\}) = k^{-1}, \quad \mu_k(\{n\}) = 0 \text{ if } n \notin \{0, k\};$$
   $$\mu(\{0\}) = 1, \quad \mu(\{n\}) = 0 \text{ if } n \neq 0.$$
   What does this example illustrate about (a) and/or (b), and why?

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Use the Fubini-Tonelli theorem to show that, if $0 < p < \infty$, then
   $$\int_X |f(x)|^p\, d\mu(x) = \int_0^\infty p t^{p-1} \varphi(t)\, dt$$
   where $\varphi(t) = \mu\{x : t < |f(x)|\}$.

3. Let $< a_k >_{k=1}^{\infty}$ and $< r_k >_{k=1}^{\infty}$ be sequences of real numbers with $\sum_{k=1}^{\infty} |a_k| < \infty$. Show that
   $$\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{|x - r_k|}}$$
   converges absolutely for almost all (with respect to Lebesgue measure) $x \in \mathbb{R}$. [*Hint:* Tackle $x \in [-n, n]$ first.]

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $< f_k >$ be a sequence of real-valued $\mathcal{A}$-measurable functions on X, and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Consider the three assertions: (S_1) $f_k \to f$ almost everywhere; (S_2) $f_k \to f$ in measure $(\lim_{k\to\infty} \mu\{x : |f_k(x) - f(x)| \geq \varepsilon\} = 0, \forall \varepsilon > 0)$;
(S_3) $f_k \to f$ in mean $\left(\lim_{k\to\infty} \int_X |f_k(x) - f(x)|\, d\mu(x) = 0\right)$.

   (a) Of the six possible implications among these, which are always true? (No proofs)
   (b) Which implications are true provided $\mu(X) < \infty$? (No proofs)
   (c) Prove an implication that is always true.
   (d) Select an implication that is true when $\mu(X) < \infty$ but may be false when $\mu(X) = \infty$. Prove it is true when $\mu(X) < \infty$, and give an example to show it may be false when $\mu(X) = \infty$.

5. Consider the measure space $([0,1], \mathcal{M}, m)$ where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and m is Lebesgue measure on $\mathcal{M}$; the word *measurable* means $\mathcal{M}$-measurable, and $||f||_p$ and L^p are defined in the usual way with respect to m, $0 < p \leq \infty$. We also define $\Phi(f) = \exp\left(\int \log|f|\, dm\right)$ provided the integral makes sense (in which case it can have any extended nonnegative real value, with the convention $\exp(-\infty) = 0$). Take as given the following facts:

   - If $f \in L^p$ then $\Phi(f)$ makes sense and $\Phi(f) \leq ||f||_p$;
   - If $f \in L^p$ and $0 < p' < p$ then $f \in L^{p'}$ and $||f||_{p'} \leq ||f||_p$;
   - If $f \in L^\infty$ then $\quad (\star) \quad \lim_{p\to 0+} ||f||_p = \Phi(f)$.

   We prove $(\star)$ if $f \in L^r$, $0 < r < \infty$. We may and shall assume $f \geq 0$.

   (a) Suppose $f \in L^1$ and $f \notin L^\infty$. Let $\gamma = \int f\, dm < \infty$; for each positive integer k let $E_k = \{x : f(x) \geq k\}$, so $mE_k > 0$, let $\gamma_k = \dfrac{\int_{E_k} f\, dm}{mE_k}$ be the average value of f on E_k, and define $g_k : [0,1] \to \mathbb{R}$ by $g_k \equiv f$ on $\tilde{E}_k$ (the complement of E_k in $[0,1]$) and $g_k \equiv \gamma_k$ on E_k. Show that if $0 < p < 1$ then $||f||_p \leq ||g_k||_p$. [*Hint:* $f^p = f^p \cdot 1$]
   (b) Continuing with notations and conventions from part (*a*), show that $\limsup_{k\to\infty} \int \log g_k\, dm \leq \int \log f\, dm$, and deduce that $(\star)$ holds for f.
   (c) We have now shown that $(\star)$ holds for any $f \in L^1$. Deduce that $(\star)$ holds for any $f \in L^r$ where r is arbitrary subject to $0 < r < \infty$.
   (d) Give an example of $f \in L^1$ such that $||f||_1 > 0$ but $\lim_{p\to 0+} ||f||_p = 0$. $\left[\textit{Hint: } \int_0^1 -\frac{1}{x}\, dx = -\infty.\right]$