Algebra preliminary exam
August 21, 2006

1. Let G be a finite group.
a) Show any subgroup of G with index 2 is a normal subgroup. (This can be useful in later parts.)
b) If G has a subgroup of index 2, show the commutator subgroup of G has even index.
c) Show a subgroup of index 2 contains all elements in G of odd order. (This is independent of part b.)
d) Use either part b or part c to prove A_4 has no subgroup of index 2.

2. Let p be an odd prime number. Prove any group of size $2p$ is either cyclic or dihedral. (That is, if $\#G = 2p$ then $G \cong \mathbf{Z}/(2p)$ or $G \cong D_p$.)

3. When G is a nontrivial finite p-group and N is a nontrivial normal subgroup of G, prove N contains a nontrivial element of the center of G. (A special case, with $N = G$, says a nontrivial finite p-group has a nontrivial center.)

4. Let I be the ideal $(3, 1+\sqrt{-5})$ in $\mathbf{Z}[\sqrt{-5}]$. Since $3\mathbf{Z} \subset I$, there is a ring homomorphism $\mathbf{Z}/3\mathbf{Z} \to \mathbf{Z}[\sqrt{-5}]/I$ given by $a \bmod 3\mathbf{Z} \mapsto a \bmod I$. Show this is an isomorphism and then show I is *not* principal.

5. Define a Euclidean domain and show $F[X]$ is Euclidean, where F is a field.

6. a) Let V be a vector space over a field F and $L\colon V \to V$ be an F-linear map. If $v_1, \dots, v_r \in V$ are eigenvectors for L with distinct eigenvalues in F, prove $v_1, \dots, v_r$ are linearly independent in V.
b) For each of the following real vector spaces V, provide an example of an $\mathbf{R}$-linear map $V \to V$ realizing the indicated vectors as eigenvectors of the linear map with distinct eigenvalues:

- $V = C^\infty(\mathbf{R})$ (the infinitely differentiable functions $\mathbf{R} \to \mathbf{R}$) and $v_i = e^{a_i x}$ for distinct real a_i,
- $V = C^\infty(\mathbf{R})$ and $v_i = \sin(a_i x)$ for distinct positive a_i,
- $V =$ all real sequences $(c_0, c_1, c_2, \dots)$ and $v_i = (1, a_i, a_i^2, \dots, a_i^n, \dots)$ for distinct real a_i.