Name: ______________________

Section: ______________________

Math 310 Preliminary Examination August 2006 ,

DO FIVE OF THE SIX QUESTIONS!

Problem 1: (20 pts)

(a) State an existence and uniqueness theorem for the equations

$$\begin{aligned} x' &= f(x,y)\,, \\ y' &= g(x,y)\,, \end{aligned}$$

with initial conditions $x(0) = a$ and $y(0) = b$ under the assumptions that f, g, and all their partial derivatives are continuous.

(b) For the system:

$$\begin{aligned} x' = {} & x(1-x-y)\,, \\ y' = {} & y(1-2x-3y)\,, \end{aligned}$$

with $x(0) = y(0) = 1/10$. Can either $x(t)$ or $y(t)$ become 0 at finite time? Justify your reasoning.

Problem 2: (20 pts)

(a) Let $\tau_0 \in (0,1)$. Find the Green's function for

$$\begin{aligned} -y'' + y &= \delta(t-\tau_0) \\ y'(0) = y(1) &= 0 \end{aligned}$$

(b) Show that there exists a unique solution for

$$\begin{aligned} -y'' + y &= \lambda \tan^{-1} y + \cos x \\ y'(0) = y(1) &= 0 \end{aligned}$$

if $|\lambda|$ is sufficiently small.

Problem 3: (20 pts)

Prove that if an operator is of the form $A = I + K$ where K is compact linear operator on a Hilbert space, then A is injective implies A is surjective.

Problem 4: (20 pts)

Find $\Delta \ln(x^2 + y^2)$ in R^2 in terms of distributional derivatives.

Problem 5: (20 pts)

Let T be a compact operator on a Hilbert space $\mathcal{H}$ and $\{\phi_n : n \in N\}$ be an orthonormal system of $\mathcal{H}$.

(a) Show that $\phi_n \rightharpoonup 0$ weakly.

(b) Using (a) or otherwise, show that $\lim_{n\to\infty} \|T\phi_n\| = 0$.

(c) Let λ_n be a sequence of complex numbers. Show that the operator S defined by $Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n$ is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.

Problem 6: (20 pts)

a) Suppose f is an operator from Banach space X to itself. Give the definition of f being Fréchet differentiable at a point $x \in X$.

b) Let $X = C[0,1]$ with sup-norm. Let $t_i \in [0,1]$ and $v_i \in C[0,1]$, and define $f(x) = \Sigma_{i=1}^{n} (x(t_i))^2 v_i$. Prove that f is Fréchet differentiable at all points of X and give a formula for f'.