MATH 313 August 2006 PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of five questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) **(a)** Let we are given $(n+1)$ points $\{x_k, y_k\}$, $\quad k = 0, 1, \ldots, n$. Show that the interpolation problem of finding a polynomial $p(x)$ whose degree does not exceed n and such that

$$p(x_k) = y_k \qquad (k = 0, 1, \ldots, n), \tag{1}$$

always has a unique solution $p(x)$.

(b) Formulate and prove the Lagrange formula for the interpolating polynomial $p(x)$ defined in (1).

(c) Let $p(x)$ again denote the interpolating polynomial defined in (1). Show that if a function $f(x)$ has an $(n+1)$st derivative, then for every argument y there exists a number s in the smallest interval $I[x_0, ..., x_n, y]$ which contains y and support abscissas x_i, satisfying

$$f(y) - p(y) = \frac{w(y) f^{(n+1)}(s)}{(n+1)!}$$

where

$$w(x) = (x - x_0)(x - x_1) \cdots (x - x_n).$$

Q2) Let the numbers a and $b(> a)$ be fixed, and let us define the scalar product in the linear space Π_n (consisting of polynomials whose degree does not exceed n) by

$$< f(x), g(x) >= \int_a^b f(x) g(x) w(x) dx, \qquad f, g \in \Pi_n,$$

where $w(x)$ is the weight function[1].

By applying the Gram-Schmidt process to $\{1, x, x^2, \ldots, x^n\}$ one obtains a system of **orthogonal** polynomials $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ such that

$$< p_k(x), p_j(x) >= 0, \qquad \text{if} \qquad k \neq j.$$

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[1] By its definition, the weight function must meet the following requirements:

- $w(x) \geq 0$ is measurable on $[a, b]$.
- All moments $\mu_k := \int_a^b x^k w(x) dx$ exist and finite for $k = 0, 1, 2, \ldots$
- For polynomials $s(x)$ which are nonnegative on $[a, b]$ the relation $\int_a^b s(x) w(x) dx = 0$ implies $s(x) = 0$ on $[a, b]$.

(a) Show that

$$< p(x), p_n(x) >= 0$$

for all $p(x) \in \Pi_{n-1}$.

(b) Show that the roots $x_1, \ldots, x_n$ of $p_n(x)$ are all real and simple, and that they all lie on the open interval (a, b).

(c) Show that the matrix

$$\begin{bmatrix} p_0(t_1) & p_0(t_2) & \cdots & p_0(t_n) \\ p_1(t_1) & p_1(t_2) & \cdots & p_1(t_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(t_1) & p_{n-1}(t_2) & \cdots & p_{n-1}(t_n) \end{bmatrix}$$

is nonsingular for the mutually distinct numbers $t_1, \ldots, t_n$.

(d) Let $w_1, w_2, \ldots, w_n$ be the solution of the nonsingular system of equations

$$\begin{bmatrix} p_0(t_1) & p_0(t_2) & \cdots & p_0(t_n) \\ p_1(t_1) & p_1(t_2) & \cdots & p_1(t_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(t_1) & p_{n-1}(t_2) & \cdots & p_{n-1}(t_n) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} < p_0(x), p_0(x) > \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Prove that the relation

$$\int_a^b w(x)p(x)dx = \sum_{k=1}^{n} w_k p(x_k)$$

holds for all polynomials $p(x) \in \Pi_{2n-1}$.

Q3) For $k = 1, 2, \ldots, n$ consider the $k \times k$ matrices

$$A_k = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 0 & \sqrt{2} \\ & & & & & \sqrt{2} & 0 \end{bmatrix}.$$

Let us define the polynomials

$$p_k(x) = det(2xI - A_k) \qquad \text{for} \qquad k = 1, 2, ...n.$$

(a) Derive the recurrence relations for $\{p_k(x)\}_{k=1}^n$ by expanding the determinants $det(2xI - A_k)$ along their top rows.

(b) By looking at the obtained recurrence relations identify polynomials$\{p_k(x)\}_{k=1}^n$ by the name (it is one of the well-known families of the orthogonal polynomials), and write down (without proving it) a well-known trigonometric expression for $p_n(x)$.

(c) Use the latter well-known trigonometric expression for $p_n(x)$ to derive (i.e., not just state it but deduce it) the formula the roots of $p_n(x)$ and thus for the eigenvalues of the matrix A_n.

Q4) (a) Prove: An $n \times n$ matrix $A = (a_{i,j})$ admits an LU-factorization $A = LU$ without pivoting and with invertible factors L and U if and only if for $k = 1, \ldots, n$, the leading principal submatrices of A of order k are all invertible.

(b) Let A be an $n \times n$ invertible matrix that admits an LU-factorization without pivoting. Show that such a factorization is unique; namely, if $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are lower triangular matrices with diag$(L1)$ =diag$(L_2) = I$ and where U_1 and U_2 are upper triangular, then $L_1 = L_2$ and $U_1 = U_2$.

(c) Suppose that A is a real $n \times n$ symmetric invertible matrix which admits an LU-factorization $A = LU$, with a lower triangular matrix L such that diag$(L) = I$, and with an upper triangular matrix U having positive diagonal entries. Show that A admits a factorization $A = \tilde{L}\tilde{L}^T$.

Q5) (a) Let $N = 2M + 1$ and consider

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M}(A_h \cos hx + B_h \sin hx) \tag{2}$$

and

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix}$$

Assume that $\Psi(x)$ and $p(x)$ agree at the N points

$$x_k = 2\pi k/N, \qquad k = 0, 1, \ldots, N-1$$

i.e.,

$$\Psi(x_k) = p(x_k), \qquad k = 0, 1, \ldots, N-1.$$

Use the relation between e^{x_k} and $e^{x_{N-k}}$ to find the matrix R such that

$$\begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_M & B_M & \cdots & B_2 & B_1 \end{bmatrix} \cdot R = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{N-1} \end{bmatrix} \tag{3}$$

(b) Explain why the matrix R in (3) is invertible, and use the uniqueness of the interpolation polynomial to show that the trigonometric polynomial (2) satisfying

$$\Psi(x_k) = y_k, \quad y_k \in \mathbb{C}, \quad k = 0, ..., N-1. \tag{4}$$

is unique.