1. Let G be a finite group acting transitively on the finite set A. Let N be a normal subgroup of G and write $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ for the distinct N-orbits in A.

   (a) For any $g \in G$ and N-orbit $\mathcal{O}_i$, show $g\mathcal{O}_i := \{ga : a \in \mathcal{O}_i\}$ is an N-orbit and the action of G on the set of N-orbits $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$, given by $g \cdot \mathcal{O}_i = g\mathcal{O}_i$, is transitive.

   (b) For $a \in A$, let $G_a = \{g \in G | ga = a\}$ be the stabilizer of a in G. Prove $r = [G : NG_a]$.

2. (a) Define a solvable group and use your definition to show every dihedral group is solvable.

   (b) Show that if $N \trianglelefteq G$ and both N and G/N are solvable, then G is solvable.

3. Let $R[G]$ be the group ring of the finite group G over a commutative ring R. Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_t$ be the different conjugacy classes of G. For each $\mathcal{K}_i$, let $C_i = \sum_{k \in \mathcal{K}_i} k$ be the sum of the members of $\mathcal{K}_i$, as an element of $R[G]$. Prove that the center of $R[G]$ is the set of sums $a_1C_1 + a_2C_2 + \cdots + a_tC_t$ for $a_1, a_2, \ldots, a_t \in R$.

4. (a) Prove that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain.

   (b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD by giving an explicit example of nonunique factorization and justifying your example.

5. (a) Give an example of a nonprincipal ideal in $\mathbb{Z}[x]$. Be sure to justify your answer.

   (b) Give an example of a nonzero prime ideal in $\mathbb{Z}[x]$ which is **not** a maximal ideal. Be sure to justify your answer.

6. Let R be a commutative ring and M be an R-module. If an R-module homomorphism $f : M \to M$ satisfies $f^2 = f$, show $M = (\ker f) \oplus f(M)$.