Math 303 Prelim

2007

1. (a) State the Radon-Nikodym theorem for finite positive measures.

(b) If ρ, ν, and μ are finite positive measures such that ν is absolutely continuous with respect to μ with Radon-Nikodym derivative f and ρ is absolutely continuous with respect to ν with Radon-Nikodym derivative g, show that ρ is absolutely continuous with respect to μ with Radon-Nikodym derivative fg.

2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose f_n is a sequence of measurable functions that converges a.e. to a function f and

$$\sup_n \int |f_n(x)|^2 \, \mu(dx) < \infty.$$

Prove that

$$\int |f_n(x) - f(x)| \, \mu(dx) \to 0.$$

3. Let f_n is a sequence of measurable functions and define

$$g_n(x) = \sup_{m \geq n} |f_m(x) - f_n(x)|.$$

Prove that if g_n tends to 0 in measure, then f_n converges almost everywhere.

4. Let $a \in (0,1)$ and let $K(x) = |x|^{-a}$ if $x \in \mathbb{R} \setminus \{0\}$. Prove that if f is non-negative and integrable with respect to Lebesgue measure on the reals, then the function g defined by

$$g(x) = \int f(y) K(x-y) \, dy$$

is finite almost everywhere. (Note that $K \notin L^p$ for any $p \in [1, \infty]$.)

5. Suppose $f : [0,1]^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-field on $[0,1]^2$ and integrable with respect to two dimensional Lebesgue measure restricted to this square.

(a) Prove that if

$$\int_0^a \int_0^b f(x, y)\, dx\, dy = 0$$

for every $a, b \in [0, 1]$, then $f = 0$ a.e.

(b) Give a counterexample to show that there exists an f such that

$$\int_0^a \int_0^1 f(x, y)\, dx\, dy = 0 \qquad \text{and} \qquad \int_0^1 \int_0^b f(x, y)\, dx\, dy = 0$$

for every $a, b \in [0, 1]$, but f is not zero almost everywhere.