Real Analysis Qualifying Exam
Summer 2008

1. State Fatou's lemma and the monotone convergence theorem, and prove that each implies the other.

2. Suppose $f_n \to f$ a.e. and f is integrable. Prove that if this is the case, then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. What if $f_n \to f$ in measure instead of a.e.?

3. (Kernel operators) Let $(X, \mathcal{M}, \mu)$ $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K \in L_2(\mu \times \nu)$. For $f \in L_2(\nu)$,
a) prove that $\int_Y |K(x,y)f(y)|d\nu(y) < \infty$ μ-a.e., and
b) with $T_K f$ defined by the formula

$$(T_K f)(x) = \int_Y K(x,y)f(y)d\nu(y), \quad \mu - a.e.,$$

show that

$$\|T_K f\|_{L_2(\mu)} \le \|K\|_{L_2(\mu\times\nu)} \|f\|_{L_2(\nu)}.$$

[This means that the linear operator T_K from $L_2(\nu)$ to $L_2(\mu)$ is continuous.]

4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $f : X \mapsto [0, \infty)$ be measurable. Let $S(t) = \mu\{x : f(x) > t\}$.
1) Prove that $f \in L_1(X, \mu)$ if and only if $S \in L_1([0, \infty), m)$, where m is Lebesgue measure, and in fact, that $\|f\|_{L_1(\mu)} = \|S\|_{L_1(m)}$.
2) Show that if $h \in L_1([a, \infty), m)$ for some $a \in \mathbf{R}$ and h is non-increasing, then $\lim_{t\to\infty} th(t) = 0$.
3) Show that if $f \in L_1(\mu)$, then $t\mu\{x : |f(x)| > t\} \to 0$ as $t \to \infty$. (Compare to Chebyshev's inequality.)

5. Let μ_k, $k \in \mathbf{N}$, be positive measures on $\mathbf{R}$.
a) Show that the set function $\mu : \mathcal{B} \mapsto \mathbf{R}^+ \cup \{0\}$ defined by

$$\mu(A) = \sum_k \mu_k(A), \quad A \in \mathcal{B}$$

is a Borel measure.
b) Assume now that $\sum_k \mu_k[-n, n]$ is finite for all n, and let $\mu_k = \lambda_k + \nu_k$ be the Lebesgue decomposition of μ_k for each k (λ_k and Lebesgue measure m are mutually singular, and ν_k is absolutely continuous w.r.t. m). Prove that if $\lambda = \sum_k \lambda_k$ and $\nu = \sum_k \nu_k$, then $\mu = \lambda + \nu$ is the Lebesgue decomposition of μ.
c) Show that if $F_k : [a, b] \mapsto \mathbf{R}^+ \cup \{0\}$ are non-decreasing, right continuous and non-negative functions, and if $F(x) := \sum_k F_k(x) < \infty$ for all x in $[a, b]$, then F is also right continuous (and, obviously, non-decreasing and non-negative) and

$$F'(x) = \sum_k F'_k(x)$$

for almost all x in $[a, b]$.