Complex Analysis Preliminary Exam

August 2009

Instructions:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii)
- The complex numbers are denoted $\mathbb{C}$, the real numbers $\mathbb{R}$, and the natural numbers $\mathbb{N}$.
- The letter z is used for complex numbers, the letters x and y are real numbers, and the letter n is a natural number.
- The open unit disc is denoted $\mathbb{D} = \{z : |z| < 1\}$.

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1. Evaluate the following integral, justifying all steps. The number a is real and positive.

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}\,dx$$

2. (i) Determine, with proof, the number of zeros of $f(z) = z^3 - z + \frac{1}{10}e^z$ in the disc $|z| < 2$.

   (ii) Suppose that $\{f_n(z)\}$ is a sequence of functions, each of which is analytic on the closure of $\mathbb{D}$ and has exactly one zero in $\mathbb{D}$. If the sequence f_n converges uniformly on the closure of $\mathbb{D}$ to a function f, must it be the case that f has exactly one zero in $\mathbb{D}$? Give a proof or a counterexample.

3. (i) Show there is no non-constant bounded analytic function on $\mathbb{C} \setminus \{0\}$.

   (ii) Show there is no non-constant bounded analytic function on $\mathbb{C} \setminus \mathbb{N}$.

   (iii) Give an example of a non-constant bounded analytic function on $\mathbb{C} \setminus [0, \infty)$.

4. Let f be analytic and satisfy $|f(z)| \leq 1$ on $\mathbb{D}$, and suppose that $f(0) = f'(0) = 0$. Prove that $|f''(0)| \leq 2$ and describe the functions having $|f''(0)| = 2$.

5. Show that the following function on $\mathbb{R}^2$ is harmonic and compute a harmonic conjugate.

$$u(x,y) = x^3 + 2xy - 2x^2 - 3xy^2 + 2y^2$$

6. Let $\mathcal{F}$ be a set of functions that are analytic on $\mathbb{D}$ and satisfy $f(0) = 1$ for all $f \in \mathcal{F}$. Let $\mathcal{F}' = \{f'(z) : f \in \mathcal{F}\}$. Show that $\mathcal{F}$ is a normal family if and only if $\mathcal{F}'$ is normal.