MATH 5510(313) August 2009 PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer either

- three out of six questions, or
- two out of three questions that belong to different areas (e.g., one on numerical linear algebra and another one on integration).

You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) **(a)** Let $x = (x_1, \ldots, x_n)^T$ be a vector whose entries are all **positive** numbers and for any vector $y \in \mathbb{R}^n$, define the quantity

$$f(y) := \inf\{\alpha > 0 \mid -\alpha x \leq y \leq \alpha x\},$$

where for two vectors $u, w \in \mathbb{R}^n$, $u \leq w$ means the every entry in w is at least equal to the corresponding entry in v. Show that $f(y)$ defines a vector norm on R^n. In the case that $x = (1, \ldots, 1)^T$, can you identify the common norm which now $f(y)$ yields?

(b) Let $A \in \mathbb{C}^{n \times n}$ and let

$$\rho(A) := \max\{|\lambda| \mid \det(A - \lambda I) = 0\}.$$

Show that the following statements are equivalent:

i) $\lim_{i \to \infty} A^i = 0$.

ii) $\rho(A) < 1$.

iii) There exists a multiplicative matrix norm $\|\cdot\|$ such that in this norm, $\|A\| < 1$.

(c) Suppose that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that there is no multiplicative norm $\|\cdot\|$ for which $\|A\| = 1$. Hence find an example of a nonmultiplicative norm.

Q2) (a) Suppose that $A = (a_{i,j})$ is an $n \times n$ invertible upper triangular matrix. Show that

$$\text{cond}_{\|\cdot\|_\infty}(A) \geq \frac{\|A\|_\infty}{\min_{1 \leq i \leq n} |a_{i,i}|},$$

where, as usual, if $\|\cdot\|$ is a matrix norm, then $\text{cond}_{\|\cdot\|}(A) = \|A\|\|A^{-1}\|$.

(b) Suppose that A is an $n \times n$ invertible matrix and that $|||\cdot|||$ is a matrix norm induced by the vector norm $\|\cdot\|$. Show that for any nonzero vector x,

$$\frac{\|x\|}{\|Ax\|} \leq |||A^{-1}|||.$$

If we use the left hand side of this inequality to obtain a lower bound on the condition number of A relative to the norm $|||\cdot|||$, how should we try to choose the vector x to get a realistic lower bound? Examplify your ideas on the matrix

$$A = \begin{bmatrix} 6 & 6 & 3.00001 \\ 10 & 8 & 4.00003 \\ 6 & 4 & 2.00002 \end{bmatrix}.$$

(c) Suppose that that $|||\cdot|||$ is a matrix norm induced by the vector norm $\|\cdot\|$. Assume that A is an $n \times n$ invertible matrix and that E is an $n \times n$ matrix for which $|||A^{-1}||| |||E||| < 1$. Show first that the matrix $A + E$ is invertible. Next, consider the linear system $Ax = b$ and its neighboring system $(A + E)x = b + \delta b$. If their exact solutions are, respectively, y and $\hat{y}$, show that

$$\frac{\|y - \hat{y}\|}{\|y\|} \leq \frac{\mathrm{cond}_{\|\cdot\|}(A)}{1 - \mathrm{cond}_{\|\cdot\|}(A)\frac{|||E|||}{|||A|||}} \left(\frac{\|\delta b\|}{\|b\|} + \frac{|||E|||}{|||A|||} \right).$$

Interpret this result numerically?

Q3) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

(a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, ...$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm of a function on the interval $[-1, 1]$.

(c) Obtain that $p(x) \approx 0.98516 + .11961x$ is the best approximation polynomial of order at most 1 to the function $f(x) = \sqrt{1 + (1/4)x^2}$ over the interval $[0, 1]$

Q4) For a given set of support points $(x_i, f_i) \quad (i = 0, 1, \dots, n)$ we denote by

$$P_{i_0, i_1, \dots, i_k}$$

that polynomial (of degree k or less) for which

$$P_{i_0, i_1, \dots, i_k}(x_{i_j}) = f_{i_j}, \qquad j = 0, 1, \dots, k.$$

(a) Prove that polynomials $P_{i_0,i_1,\ldots,i_k}$ are linked by the following recursion

$$p_i(x) = f_i,$$

$$P_{i_0,i_1,\ldots,i_k}(x) = \frac{(x-x_{i_0})P_{i_1,i_2,\ldots,i_k}(x) - (x-x_{i_k})P_{i_0,i_1,\ldots,i_{k-1}}(x)}{x_{i_k}-x_{i_0}}$$

(b) Let

$$P_{i_0,i_1,\ldots,i_k}(x) = f_{i_0} + f_{i_0,i_1}(x-x_{i_0}) + \ldots + f_{i_0,i_1,\ldots,i_k}(x-x_{i_0})(x-x_{i_1})\cdot\ldots\cdot(x-x_{i_{k-1}})$$

be a Newton representation of $P_{i_0,i_1,\ldots,i_k}(x)$, where the coefficients are called *divided differences*. Prove that divided differences satisfy the following recursion

$$f_{i_0,i_1,\ldots,i_k} = \frac{f_{i_1,i_2,\ldots,i_k} - f_{i_0,i_1,\ldots,i_{k-1}}}{x_{i_k}-x_{i_0}}$$

(c) Prove the following theorem (error in polynomial interpolation).
If the function f has an $(n+1)$st derivative, then for every argument $\bar{x}$ there exist a number ξ (in the smallest interval containing $x_{i_0}, x_{i_1}, \ldots x_{i_n}, \bar{x}$), satisfying

$$f(\bar{x}) - P_{i_0,i_1,\ldots,i_n}(x) = \frac{w(\bar{x})f^{(n+1)}(\xi)}{(n+1)!},$$

where

$$w(x) = (x-x_{i_0})(x-x_{i_1})\ldots(x-x_{i_n}).$$

Q5) The integration formulas of Newton and Cotes are obtained if the integrand is replaced by a suitable interpolating polynomial $P(x)$ and then if the value $\int_a^b P(x)dx$ is taken as approximate value for $\int_a^b f(x)dx$. The following questions are concerned with Newton-Cotes formulas.

(a) Derive the trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}[f(a)+f(b)].$$

as a special case of Newton-Cotes formula in which $P(x)$ is a linear interpolating polynomial using the two points $(a, f(a))$ and $(b, f(b))$.

(b) It is well-known that the error of trapezoidal rule

$$I_i = \frac{h}{2}[f(x_i)+f(x_{i+1})] \qquad \text{(here } h = x_{i+1}-x_i)$$

on the interval (x_i, x_{i+1}) is as follows

$$I_i - \int_{x_i}^{x_{i+1}} f(x)dx = \frac{h^3}{12}f^{(2)}(\xi_i), \qquad \xi_i \in (x_i, x_{i+1})$$

Use the above result to derive the formula for the error

$$T(h) - \int_a^b f(x)dx = \frac{(b-a)}{12}h^2 f^{(2)}(\xi), \qquad \xi \in (a,b)$$

in the *composite* trapezoidal rule

$$T(h) = h[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{f(b)}{2}]$$

(c) Derive the Simpson's rule

$$\int_a^b f(x)dx \approx \frac{(b-a)}{3}(f(a) + 4f(\frac{a+b}{2}) + f(b)).$$

as a special case of Newton-Cotes formula in which $P(x)$ is a quadratic interpolating polynomial using the three points $(a, f(a))$ and $(\frac{a+b}{2}, f(\frac{a+b}{2}))$ and $(b, f(b))$.

(d) It is well-known that the error of the Simpson's rule on the interval (x_i, x_{i+2}) (here $h = x_{i+1} - x_i$) is as follows

$$I_i - \int_{x_i}^{x_{i+2}} f(x)dx = \frac{h^5}{90} f^{(4)}(\xi_i), \qquad \xi_i \in (x_i, x_{i+2})$$

Use the result to derive the formula for the error

$$S(h) - \int_a^b f(x)dx = \frac{(b-a)}{180} h^4 f^{(4)}(\xi), \qquad \xi \in (a, b)$$

in the *composite* Simpson's rule

$$S(h) = \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \ldots + 2f(b-2h) + 4f(b-h) + f(b)]$$

Q6) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Define a Hankel matrix. Let H be an $n \times n$ positive definite Hankel matrix. Relate the factorization

$$H\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of H to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let the above Hankel matrix H be a moment matrix, i.e., $H = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \tag{2}$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Derive a algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its two "predecessors" u_{k-2}, u_{k-1} $(k = 3, \ldots, n)$.

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.