Measure and Integration (Math 5111) Aug 2009
Qualifying Exam

ID:________________
Last: ________________ **First:** ________________

- Do Problem 5 and any three (complete, not combinations of parts) of Problems 1-4. Mark on this form the problem NOT to be graded.
- Results proved in the textbook/s (Folland, Royden, Rudin or Dudley) should be applied without proof. In this case you have to state accurately the result you are using.
- Below m denotes the Lebesgue measure on $\mathbb{R}$.

1. (a) Suppose that $\{\mu_n : n \in \mathbb{N}\}$ is a sequence of σ-finite measures on the measurable space $(X, \mathcal{F})$. For $A \in \mathcal{F}$ let $\nu(A) = \sum_{n=1}^{\infty} \mu_n(A)$. Prove that ν is a measure on $(X, \mathcal{F})$.

   (b) Let ν be the measure from part (a). Prove that one can write $\nu = \sum_{n=1}^{\infty} \rho_n$, where $\{\rho_n : n \in \mathbb{N}\}$ is a sequence of measures on $(X, \mathcal{F})$ satisfying all of the following

   (i) For all $n \in \mathbb{N}$, $\rho_n \ll \mu_n$;

   (ii) If $n > 1$, then $\rho_k \perp \rho_n$ for all $k < n$;

   (iii) For all $n \in \mathbb{N}$, $\frac{d\rho_n}{d\nu} \in \{0, 1\}$, ν-almost everywhere.

2. (a) Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose that $\{f_n : n \in \mathbb{N}\}$ is a sequence of functions in $L^1(X, \mathcal{F}, \mu)$ which satisfy $|f_n| \leq h$ for some $h \in L^1(X, \mathcal{F}, \mu)$ and $\lim_{n\to\infty} f_n = f$ in measure. Prove that $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$.

   (b) Suppose $f : [0, \infty) \to [0, \infty)$ is a three times differentiable nonnegative function, which satisfies $f(0) = f'(0) = 0$. Assume further that $f''(0) > 0$ and $f^{(3)} \geq 0$. Prove

$$\lim_{M\to\infty} \sqrt{M} \int_{[0,\infty)} e^{-Mf(x)} dm = \frac{c}{\sqrt{f''(0)}}, \text{ where } c = \int_{[0,\infty)} e^{-\zeta^2/2} dm(\zeta) = \sqrt{\frac{\pi}{2}}.$$

3. Let $f \in L^1(X, \mathcal{F}, \mu)$ and assume further $\mu(X) < \infty$. Consider the function

$$g(c) = \int |f(x) - c| d\mu.$$

   (a) Prove that g is absolutely continuous on $\mathbb{R}$ and $\lim_{|c|\to\infty} g(c) = \infty$.

   (b) Find $g'(c)$, and prove that $g(c_0) = \min_{c\in\mathbb{R}} g(c)$ if and only if $\mu(\{x : f(x) < c_0\}) = \mu(\{x : f(x) > c_0\})$ (such a c_0 is called a median).

4. (a) Assume that f an absolutely continuous function on $\mathbb{R}$ satisfying $f(0) = 0$ and $f' \in L^p(m)$ for some $p > 1$. Prove that for all $g \in L^q(m)$, we have

$$\int_0^1 |fg| dm \leq \left(\frac{1}{p}\right)^{1/p} \left(\int_0^1 |f'|^p dm\right)^{1/p} \left(\int_0^1 |g|^q dm\right)^{1/q}.$$

   Here q is the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$.

   (Hint: use the assumptions on f to express it in terms of f')

   (b) State and prove the analog of the inequality of part (a) for the case $p = 1$.

5. True/False. Determine whether each of the above is true or false (not always true). In the former case, prove. In the latter case, provide a counterexample.

   Note: the parts are not related.

   (a) The function
   $$f(x) = \begin{cases} \frac{1}{q} & x = p/q,\ p \in \mathbb{Z}_+, q \in \mathbb{N} \text{ are relatively prime} \\ 0 & \text{otherwise.} \end{cases}$$
   is Riemann integrable on $[0, 1]$.

   (b) Suppose f is a measurable function on the product measure space $(X \times Y, \mathcal{F} \times \mathcal{G}, \mu \times \nu)$, and that both iterated integrals $\int_Y \int_X f(x, y) d\mu(x) d\nu(y)$, $\int_Y \int_X f(x, y) d\nu(y) d\mu(x)$ are well defined, finite and equal to 0. Then $f \in L^1(\mu \times \nu)$.

   (c) Suppose that the sequence of measurable functions $\{f_n : n \in \mathbb{N}\}$ satisfies $\lim_{n\to\infty} \int f_n g dm = 0$ for all $g \in L^1(m)$. Then $\lim_{n\to\infty} f_n = 0$ in measure.

   (d) Suppose that $(X, \mathcal{F}, \mu)$ is a finite measure space. Then $1 \leq p_1 < p_2$ implies $L^{p_1}(\mu) \supset L^{p_2}(\mu)$.

   (e) If $\{f_n : n \in \mathbb{N}\}$ is a sequence of nonnegative Lebesgue-measurable functions then $\limsup_{n\to\infty} \int f_n dm \leq \int \limsup_{n\to\infty} f_n dm$.