August 12, 2011

2011 Complex Prelim

Δ is the open unit disk; ∂D is the boundary of the domain D; $\mathbb{C}$ is the complex plane and $\hat{\mathbb{C}}$ is the Riemann sphere (i.e. the extended complex plane); $\mathcal{O} = \mathcal{O}(D)$ is the set of function holomorphic (or equivalently analytic) in the domain D.

Justify your reasoning in all problems.

(1) Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$

This problem requires more than just a calculation. Justify all steps, including all the estimates and taking limits that are needed.

(2) Suppose

$$D := \{z \in \mathbb{C} : |z-i| < \sqrt{2} \text{ and } |z+i| < \sqrt{2}\}.$$

(a) Prove that there is no conformal map of D onto $\mathbb{C}$.
(b) If there is one, find a conformal map of D to the unit disk Δ. If none exists, prove it.

(3) Prove that the Fundamental Theorem of Algebra is a direct consequence of Rouché's Theorem.

(4) (a) Suppose f is holomorphic in $\Delta \setminus \{0\}$. What consequence would follow if $|z^2 f(z)|$ is bounded?
(b) Consider a family of functions that are holomorphic on $\Delta \setminus \{0\}$ and for which there is a uniform bound for $|z^2 f(z)|$ on $\Delta \setminus \{0\}$. Is this family normal?

(5) Consider all functions f which are holomorphic for $Re(z) > 0$, take values in Δ, and vanish at $z = 1$. What is the least upper bound for $|f(2)|$? Is it achieved?