5310 PRELIM
Introduction to Geometry and Topology
August 2011

Justify all your steps rigorously. You may use any results that you know, unless the question says otherwise, or unless the question asks you to prove essentially the same result.

1. Prove that if a space X is path-connected, then it is also connected.

2. Let $D \subset \mathbb{R}^2$ be the closed unit disc. Define an equivalence relation on D by
$$(x, y) \sim (\bar{x}, \bar{y}) \Leftrightarrow x^2 + y^2 = \bar{x}^2 + \bar{y}^2.$$
Prove that the quotient is homeomorphic to the unit interval: $D/\sim \cong [0, 1]$

3. Let X be a topological space, and $\sim$ an equivalence relation on X. Decide whether the following statements are true:

   (a) If X is compact, then so is $X/\sim$.

   (b) If $X/\sim$ is compact, then so is X.

   (c) If X is Hausdorff, then so is $X/\sim$.

   (d) If $X/\sim$ is Hausdorff, then so is X.

   For each statement, either give a counter-example, or give a proof.

4. (**Path lifting property.**) Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a base-point preserving covering map. Given any path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x_0$, show that there exists a lift $\tilde{\gamma}\colon [0, 1] \to \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$.

5. Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a base-point preserving normal covering. *(Note: "normal coverings" are also sometimes called "regular coverings".)* Let a, b be two loops in X, based at x_0. Let $\tilde{a}$ be the lift of a, starting at $\tilde{x}_0$; similarly, let $\widetilde{bab^{-1}}$ be the lift of bab^{-1} starting at $\tilde{x}_0$.

   Show that $\tilde{a}$ is a loop if and only if $\widetilde{bab^{-1}}$ is a loop.

6. Let X be the plane $\mathbb{R}^2$ with two points removed: $X = \mathbb{R}^2 \setminus \{(1, 0), (-1, 0)\}$. Prove that $\pi_1(X) = \mathbb{Z} \star \mathbb{Z}$.