Abstract Algebra Prelim Aug. 2012

1. Let G_1 and G_2 be finite groups of order n_1 and n_2. If n_1 and n_2 are relatively prime, prove every subgroup of $G_1 \times G_2$ has the form $H_1 \times H_2$, where H_i is a subgroup of G_i.

2. Let G be a finite group acting on a set X. For a point $x \in X$, prove there is a bijection between the G-orbit of x and the left coset space G/H_x, where $H_x = \{g \in G : gx = x\}$ is the stabilizer subgroup of x.

3. Let R be a commutative ring. The two parts of this question, about ideals in R, do not depend on each other.
   (a) Let I and J be ideals in R, and let P be a prime ideal in R. If $IJ \subset P$, prove $I \subset P$ or $J \subset P$. (Recall the product ideal IJ is the ideal in R generated by all products xy with $x \in I$ and $y \in J$.)
   (b) Let P_1, P_2, and P_3 be prime ideals of R. If an ideal I satisfies $I \subset P_1 \cup P_2 \cup P_3$, then prove $I \subset P_i$ for some i. (Hint: Start by assuming I is not in $P_1 \cup P_2$, $P_1 \cup P_3$, or $P_2 \cup P_3$.)

4. State Zorn's lemma and then use it to prove every nonzero commutative ring contains a maximal ideal.

5. State the class equation for a finite group G and use it to show that every group of order 32 must have a non-trivial center.

6. Give examples as requested, with brief justification.
   (a) A permutation $\pi \in S_4$ such that $(243) = \pi(123)\pi^{-1}$.
   (b) A prime ideal in $\mathbf{Z}/20\mathbf{Z}$.
   (c) A unit other than ± 1 in $\mathbf{Z}[\sqrt{11}]$.
   (d) An integer $m > 1$ such that the group $(\mathbf{Z}/m\mathbf{Z})^\times$ is not cyclic.