Math 5410 Preliminary Exam

Aug 2012

Name Signature______________________________

Do all 5 problems.

1. State definition of compact operator defined on a Banach space to itself.
   (a) Give an example of compact linear operator defined on l^2 and an example of NON compact linear operator defined on l^2. Explain why.
   (b) Let T be a compact linear operator on a Hilbert space H. Prove that if $I+T$ is injective, then $I+T$ is surjective.
2. Find a function in $C^1[0,1]$ that minimizes $\int_0^1 (u')^2 + u^2 dt$ with constraints $u'(0) = u(1) = 0$.
3. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let K be a nonempty closed convex set in H.
   (a) Prove $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all x, y in H.
   (b) Prove that for any $x \in H$, there is a unique $y \in K$ such that $\|x-y\| = \text{disc}(x, K)$.
   (c) Let $x \in H$ and $y \in K$ be the closest point to x, show that $\langle x-y, v-y \rangle \leq 0$ for all $v \in K$.
4. Suppose f is an operator defined on a Banach space X to itself.
   (a) State the definition of f being Fréchet differentiable at a point x in X.
   (b) Define $f; C[0,1] \to C[0,1]$ by $f(x)(t) = x^2(t) + \int_0^1 x^2(st)\, ds$. Determine whether f is Fréchet differentiable and find f' if it is differentiable.
5. Let $U(x,y)$ be the characteristic function of the first quadrant in $xy-$plane. Find the distributional derivative $\frac{\partial^2 U}{\partial x \partial y}$.