Preliminary Exam in Complex Analysis
August 2012

Instructions

All assertions require written justification. In particular, state and verify the hypotheses of any theorems you use. In complex analysis the terms 'analytic' and 'holomorphic' are used interchangeably.

In a multipart problem, if you can't do an earlier part of the problem you may nevertheless assume it when attempting a later part.

1. Suppose the function f is analytic in a simply connected region Ω. Show that there exists an analytic function F in Ω so that $F'(z) = f(z)$ for all $z \in \Omega$.

2. Let P and Q be polynomials with $\deg Q \geq \deg P + 2$ (here deg stands for degree). Show that there is a number $r > 0$ so that if γ is a closed curve outside the disc of radius r centered at the origin, then $\displaystyle\int_\gamma \frac{P(z)}{Q(z)}\, dz = 0$. (If you don't see how to proceed, you might first want to do this assuming that γ is a circle.)

3. Prove that if f and g are entire functions and $(f \circ g)(z)$ is a polynomial, then both f and g are polynomials.

4. Prove Vitali's Theorem:
Suppose the sequence of analytic functions $\{f_k\}$ is locally uniformly bounded in a region Ω, there is a sequence $\{a_n\}$ in Ω so that $\lim_{n\to\infty} a_n = a \in \Omega$ and for each n the limits $\lim_{k\to\infty} f_k(a_n)$ exist .

(a) Show that the sequence $\{f_k\}$ converges pointwise everywhere in Ω.

(b) Show that the sequence $\{f_k\}$ converges uniformly on compact subsets in Ω.

5. Let Δ denote the unit disc centered at the origin and $\overline{\Delta}$ its closure. A function is analytic on a closed set if it is analytic in a domain containing the set.

(a) Prove that an analytic function $f : \overline{\Delta} \to \Delta$ has exactly one fixed point; that is, exactly one point $z_0 \in \overline{\Delta}$ so that $f(z_0) = z_0$.

(b) Suppose $z_0 \in \overline{\Delta}$ is the fixed point of f from part (a). Given $z \in \Delta$ define the sequence $z_1 = f(z), z_2 = f(z_1), \ldots, f(z_n) = z_{n+1} \ldots$ Prove that for any $z \in \Delta$, $\lim_{n\to\infty} z_n = z_0$. (If you are stumped, try the case where $z_0 = 0$.)

(c) Suppose $f : \Delta \to \Delta$ is analytic and f is not the identity function $f(z) = z$. Is it possible for f to have no fixed points or more than one fixed point?