Thursday, August 23, 2012

2012 Real Analysis Prelim Exam

Justify your reasoning in all problems.

(1) (a) Does there exist a measure μ on the set of rational numbers $\mathbb{Q}$ such that all intervals are measurable and $\mu([0,q)) = q$ for any positive rational number q? Prove or disprove.

(b) Suppose $(\mathbb{R}, \mathcal{A}, \mu)$ is the measure space where $\mathcal{A}$ is the σ-algebra of all subsets and μ is the counting measure (which means that $\mu(A) = |A|$, the cardinality of A, if A is a finite set, and $\mu(A) = \infty$ if A is an infinite subset). Prove or disprove that the function

$$F(x) = \begin{cases} e^{-|x|} & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is integrable.

(c) in the same situation as in (b), prove or disprove that finitely supported functions are dense in $L^1(\mathbb{R}, \mathcal{A}, \mu)$.

(2) Suppose $(\mathbb{R}, \mathcal{A}, \mu)$ is a measure space. Prove that

$$\int_X f_n \, d\mu \to \int_X f \, d\mu$$

if f_n, g_n, f, g are integrable, $f_n \to f$ and $g_n \to g$ μ-a.e., $|f_n| \leqslant g_n$ for all n, and $\int_X g_n \, d\mu \to \int_X g \, d\mu$.

Hint: use Fatou's lemma for $g_n + f_n$ and $g_n - f_n$, or for $2g_n - |f_n - f|$.

(3) Prove that $F(x) = \sum\limits_{n=0}^{\infty} e^{-n} \cos(1 + n^2 x^2)$ is a differentiable function on $\mathbb{R}$.

(4) (a) Suppose $(\mathbb{R}, \mathcal{A}, \mu)$ is a measurable space where $\mathcal{A}$ is the Borel σ-algebra of subsets and μ is a Lebesgue-Stieltjes measure which is translation invariant in the sense that $\mu(A) = \mu(\{x : x + y \in A\})$ for any Borel set A and any $y \in \mathbb{R}$. Prove that if μ is finite on bounded intervals, then it is a multiple of the Lebesgue measure.

(b) Does there exist a non-zero $f \in L^1([0,\infty), \mathcal{A}, \lambda)$ such that $\int_{[0,q]} f \, d\lambda = 0$ for any positive rational number q? Here $([0,\infty), \mathcal{A}, \lambda)$ is the measure space with the Borel σ-algebra $\mathcal{A}$ and the Lebesgue measure λ. Prove or disprove.

(5) State and prove the Minkowski inequality.