Name: ______________________________

Math 5410 Prelim August, 23, 2013

(1) State and prove an existence theorem for the equation $\frac{dx}{dt} + f(x) = 0$ with initial conditions $x(0) = 0$ and $x'(0) = 0$ under the assumption that f is continuous and $| f | \leq r$. (You can assume Rothe's fixed point theorem.).

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au = u'' - u$$

with $u'(0) = u(1) = 0$.
If $Au = f(x)$, express the function u in terms of G and f.

(2b) Define $T : L^2(0, 1) \to L^2(0, 1)$ such that for any $f \in L^2(0, 1)$,

$$(Tf)(x) = \int_0^1 G(x, y)f(y)\, dy \ .$$

Explain what the spectral theorem is and why it is applicable.

(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3) Let

$$U(x, y) = \ln(x^2 + y^2)$$

Compute distributionally $\Delta U = (\partial_x^2 + \partial_y^2)U$ in $\Re^2$.

(4) Let H be a Hilbert space and $K : H \to H$ is a linear, bounded, compact operator. Define $A = I + K$. Show that if A is surjective, then it is injective.

(5) Let $J : H_0^1(\Omega) \to \Re$ be defined by

$$J(u) = \int_\Omega (| \nabla u |^2 /2 + u^4/4 - hu)dv$$

for h fixed in $L^2(\Omega)$ where Ω is a bounded region in $\Re^3$.
(a) Find the Frechet derivative of J.
(b) Show that $\inf J$ is attained. You may use the fact that $H_0^1(\Omega)$ is compactly embedded in $L^t(\Omega)$for $t < 6$ and that $\int_\Omega \mid \nabla u \mid^2 dV$ is a norm on $H_0^1(\Omega)$