5310 PRELIM

Introduction to Geometry and Topology

August 2013

You may use any result that has been proven in class, unless the question directly asks you to prove the result. Please, **state** the results you are using, and **check** that the assumptions are satisfied.

1. Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Compute $\pi_1(X)$.

2. Suppose a space X has a compact universal covering $\widetilde{X}$. Show that the fundamental group of X is finite.

3. Consider the following equivalency relation on $\mathbb{R}$: $x \sim y$ if $x - y$ is rational. Let X be the quotient space $\mathbb{R}/\sim$ with the quotient topology. Describe the topology of X as explicitly as you can, and decide if X is Hausdorff. Justify your answer.

4. Let $\mathbb{D}$ be the closed unit disk in the plane and $f : \mathbb{D} \to \mathbb{D}$ is a homeomorphism. Show that f maps the unit circle $\mathbb{S}^1 \subset \mathbb{D}$ into itself.

5. (a) Prove that if a path connected, locally path connected space X has finite fundamental group, then any map $f : X \to \mathbb{S}^1$ is homotopic to a constant map (null-homotopic).

   (b) Prove that any map $f : \mathbb{RP}^2 \to \mathbb{T}^2$, from the real projective plane to the torus, is homotopic to a constant map (null-homotopic).

6. Think of $\mathbb{S}^3$ as the unit sphere in $\mathbb{C} \oplus \mathbb{C}$ and $\mathbb{S}^2$ as the unit sphere in $\mathbb{C} \oplus \mathbb{R}$. With respect to these coordinates, define $h : \mathbb{S}^3 \to \mathbb{S}^2$ by $h(z, w) := \left(2z\bar{w}, |z|^2 - |w|^2\right)$.

   (a) Show that h is a quotient map.

   (b) Identify $\mathbb{S}^1$ with the unit circle in $\mathbb{C}$. Define an action of $\mathbb{S}^1$ on $\mathbb{S}^3$ by $a \cdot (z, w) := (az, aw)$, for any $a \in \mathbb{S}^1 \subset \mathbb{C}$, where the multiplication is in $\mathbb{C}$. Prove that the orbits of this action coincide with the fibers of the function h, i.e. the preimages of points in $\mathbb{S}^2$ under h.