Abstract Algebra Prelim Aug. 2014

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. a) Define Sylow subgroups of a finite group and state the Sylow theorems.

   b) Use the Sylow theorems to prove that every group of order 35 is cyclic.

2. Let G be a group containing an infinite cyclic normal subgroup $H = \langle \alpha \rangle$ such that the quotient group $G/H = \langle \overline{\beta} \rangle$ is also infinite cyclic. Prove there are two possibilities for G up to isomorphism, one abelian and one non-abelian.

3. a) State Zorn's Lemma.

   b) Use Zorn's lemma to prove that in any nonzero commutative ring R the intersection $\bigcap \mathfrak{p}$ of all the prime ideals of R is the set of nilpotent elements in R.

4. a) Show that the only units in $\mathbf{Z}[\sqrt{-5}]$ are ± 1.

   b) Prove that $\mathbf{Z}[\sqrt{-5}]$ is not a unique factorization domain. (Hint: $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.)

5. Prove that $\mathbf{Z}[x]$ has unique division with remainder if the divisor is monic: if $f(x)$ and $g(x)$ are in $\mathbf{Z}[x]$ and $g(x)$ is monic, then there are unique $q(x)$ and $r(x)$ in $\mathbf{Z}[x]$ such that (i) $f(x) = g(x)q(x) + r(x)$ and (ii) $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

6. Give examples as requested, with brief justification.

   (a) A group G and a subgroup H such that there are two elements of H that are not conjugate in H but are conjugate in G.

   (b) Three distinct groups $H \subset K \subset G$ such that $H \lhd K$ and $K \lhd G$, but H is not normal in G.

   (c) A generating set for the ideal $\{f(X) \in \mathbf{Z}[X] : f(2) \equiv 0 \bmod 5\}$ in $\mathbf{Z}[X]$.

   (d) A nontrivial character of $(\mathbf{Z}/11\mathbf{Z})^\times$ that is trivial at $-1 \bmod 11$.