Numerical Analysis July 24, 2014

Qualifying Exam

1. Let $A = xx^T$ for $x \in \mathbb{R}^N$, $x \neq 0$. Show that the rank of the matrix A is 1.

2. Determine the polynomial of degree at most $n-1$ which best approximates the polynomial

$$Q(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

on the interval $[a, b]$ and show that its maximum deviation from Q is given by

$$\frac{1}{2^{n-1}} \left(\frac{b-a}{2} \right)^n a_0.$$

3. Consider the function $f(x) = e^{\lambda x}$, $\lambda \in \mathbb{R}$, on an interval $[a, b]$. Show that the error $f - p_n$ for the Lagrange interpolation of f over uniformly distributed $n+1$ points from $[a, b]$ uniformly converges to zero as $n \to \infty$, i.e.

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \to 0 \quad \text{as} \quad n \to \infty, .$$

What distinguishes this function from $f(x) = (1 + x^2)^{-1}$, for which the Lagrange interpolation does not converge uniformly as $n \to \infty$?

4. Derive a Gaussian quadrature formula such that the integral

$$I = \int_{-1}^{1} p(x) \sqrt{|x|} dx$$

computes exactly for all cubic polynomials $p(x)$.

5. Consider the linear system

$$Ax = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

How big the relative errors $\|\delta x\|_1 / \|x\|_1$ and $\|\delta x\|_\infty / \|x\|_\infty$ can be when the relative error of the matrix is at most $\pm 1\%$ and the right hand side $\pm 3\%$? (Hint: Compute the inverse of the matrix and determine the 1-and ∞-condition numbers, i.e. $\kappa_1(A)$ and $\kappa_\infty(A)$.)

6. Show that if f is twice continuously differentiable, then the Newton's method converges locally quadratically towards the root $\bar{x}$, i.e.

$$\frac{\|x_{n+1} - \bar{x}\|}{\|x_n - \bar{x}\|^2} \leq C \quad \text{as} \quad n \to \infty.$$

Show that if f is only once continuously differentiable, then the Newton's method converges locally super-linearly towards the root $\bar{x}$, i.e.

$$\frac{\|x_{n+1} - \bar{x}\|}{\|x_n - \bar{x}\|} \to 0 \quad \text{as} \quad n \to \infty,$$

i.e. faster than simple fixed point iteration.