Prelim Exam Applied Math

August 2015

Name____

Instructions. You have four hours for this exam.

1. Find the Green's function for

$$\begin{cases} x'' + x' - 2x = y \\ x(0) = 0 = x(1) \end{cases} .$$

2. Let K be a closed convex set in a Hilbert space X.

   (a) For any $x \in X$, show there exists a unique $y \in K$ that is closest to x.

   (b) Show that $\mathcal{R} \langle x - y, v - y \rangle \leq 0$ for $\forall\ v \in K$. Here $\mathcal{R}$ indicates the real part.

   (c) Write $y = Px$, show that $\|Px - Pz\| \leq \|x - z\|$ for any $x, z \in K$.

3. Let L be a bounded linear operator defined on a real Hilbert space X. Define $F(x) = \langle x, Lx \rangle$.

   (a) Give the definition of Frechét derivative.

   (b) Determine whether F is differentiable at x and find $F'(x)$ if it exists.

4. Let $\chi_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$ be the characteristic function of a set A.

(a) Find the distributional derivative of χ_A for $A = (a, b)$.

(b) Let $f = \frac{\chi_{(a,b)}}{b-a}$, and $f_j(x) = j^n f(jx)$ for $j = 1, 2 \cdots$, show $\widetilde{f_j} \to \delta$. Here $\widetilde{f_j}(\varphi) = \int f_j(x) \varphi(x)$ is the distribution induced by f_j.

5. Let $\{\mathbf{e}_1, \mathbf{e}_2 \cdots\}$ be an orthonormal sequence in a Hilbert space X. Let $Ax = \sum \lambda_n \langle x, \mathbf{e}_n \rangle \mathbf{e}_n$ where $\sup |\lambda_n| < \infty$.

   (a) Prove the series defining Ax converges.

   (b) Prove A is bounded.

   (c) Prove A is compact if and only if $\lambda_n \longrightarrow 0$.

6. Let X be a linear space and A: $X \to X$ be a linear transformation. Assume A is surjective but not injective, then $\ker A^n$ is a proper subset of $\ker A^{n+1}$ for all n.