COMPLEX ANALYSIS PRELIM, AUGUST 2015

Instructions

- Clearly state and verify the hypotheses of any theorems you use.
- That *prove or disprove a statement* means either prove the statement or provide a counter example.
- The terms "holomorphic" and "analytic" are used interchangeably.
- Denote $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C}; |z| \leq 1\}$. Let A and B be two sets. Denote $A \setminus B = \{x \in A; x \notin B\}$.

1. Let Ω be the interior of the square with vertices at the points 1, i, -1, and $-i$. Show that there is a one-to-one holomorphic map $h : \Omega \to \mathbb{D}$ satisfying $h(0) = 0$, and $h'(0) > 1$.

2. Show that an entire function f is a nonconstant polynomial if and only if f is *proper*, i.e., for every constant $M > 0$, the set $\{z; |f(z)| \leq M\}$ is compact.

3. Prove or disprove the statement: Let $G = \mathbb{D} \setminus \{0, i/2\}$. Then, the holomorphic automorphism group $\mathrm{Aut}(G)$ consists of the identity map only; that is, if a holomorphic map $f : G \to G$ is one-to-one and onto, then $f(z) = z$ for all $z \in G$.

4. Evaluate the integral $\int_{-\infty}^{\infty} e^{ipx - qx^2} dx$, where $p, q \in \mathbb{R}$ and $q > 0$. Justify your reasoning. (You can use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ without justification.)

5. Let g be a holomorphic function on $\mathbb{D} \setminus \{0\}$, and denote $g_m(z) = g(z/m)$ for each positive integer m. Suppose that $\{g_m\}_{m=1}^{\infty}$ has a subsequence $\{g_{m_k}\}_{k=1}^{\infty}$ which is uniformly bounded by 1 on the circle $\{z; |z| = 1/2\}$, i.e.,
$$\max_{|z|=1/2} |g_{m_k}(z)| \leq 1 \quad \text{for all } k \geq 1.$$
Show that g can be extended to a holomorphic function on $\mathbb{D}$.

6. Prove or disprove the statement: Let f be a holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose that for a constant $0 < \delta < 1/10$, $f(e^{i\theta}) = 1 + 2i$ for all $-\delta\pi < \theta < \delta\pi$. Then $f \equiv 1 + 2i$ on $\overline{\mathbb{D}}$.