Probability Prelim Exam for Actuarial Students
August 2015

1. (10 points) Let X and Y be independent random variables. Let $f, g : \mathbf{R} \to \mathbf{R}$ be Borel-measurable functions. Show that the random variables $f(X)$ and $g(Y)$ are independent.

2. (20 points) Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $A_n \in \mathscr{F}$ for $n = 1, 2, \ldots$. Show that

   (a) (10 points) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then
   $$P\left(\limsup_n A_n\right) = 0.$$

   (b) (10 points) If $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent, then
   $$P\left(\limsup_n A_n\right) = 1.$$

3. (10 points) Let X be a non-negative random variable on a probability space $(\Omega, \mathscr{F}, P)$. Show that for all $\alpha > 0$,
$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}.$$

4. (10 points) Let $X, X_1, X_2, \ldots, Y_1, Y_2, \ldots$ be random variables on a defined probability space. Show that if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$, for some constant c, then
$$X_n + Y_n \xrightarrow{d} X + c,$$
where $\xrightarrow{d}$ means convergence in distribution.

5. (10 points) Let X and Y be two random variable in a probability space $(\Omega, \mathscr{F}, P)$. Suppose that X and Y are independent. Show that
$$E(Y|X) = E(Y), \quad a.s.$$

6. (10 points) Let $\{X_n\}$ be a submartingale. Let τ_1 and τ_2 be stopping times. Suppose that $0 \leq \tau_1 \leq \tau_2 \leq M$ almost surely for some positive integer M. Show that
$$E(X_{\tau_2}) \geq E(X_{\tau_1}).$$