Real Analysis Preliminary Exam, August 2015

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}$ is denoted by m or dm, or by dx or dy. The complement of a set E is denoted by E^c.

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1. (10 points) Compute
$$\int_0^\infty \int_0^{\sqrt{\pi}/2} e^{-y/x} \cos(x^2)\, dx\, dy.$$
Justify all steps in your computation.

2. (10 points) Suppose $h(x) \in L^p(m)$ on $\mathbb{R}$ and $1 < p < \infty$. Prove that, when $q < (p-1)/p$,
$$I(h) = \int_0^1 \frac{h(x)}{x^q}\, dx < \infty.$$
Also, for any $q \geq (p-1)/p$ give an example of a non-negative function $h \in L^p$ such that $I(h) = \infty$.

3. (10 points) Let f and $\{f_n\}_{n=1}^\infty$ be measurable functions, and suppose that for any $\epsilon > 0$ we have
$$\sum_1^\infty \mu\{x : |f_n(x) - f(x)| > \epsilon\} < \infty.$$
Prove that f_n converges to f μ-a.e.

4. (20 points) Let f be a Lebesgue integrable function on $\mathbb{R}$ such that $\int_I f\, dm = 0$ whenever I is an open interval.
   (i) Prove that $\int_U f\, dm = 0$ if U is any open set.
   (ii) Show that $\int_E f\, dm = 0$ if E is any Lebesgue measurable set.
   (iii) Show that $f = 0$ a.e. with respect to m.
   (iv) Let g be an integrable function supported on $[0,1]$, and suppose that for all $k \in \mathbb{N} \cup \{0\}$,
   $$\int x^n g(x)\, dx = 0$$
   Prove that $g = 0$ a.e. with respect to Lebesgue measure. *(Hint: approximate the characteristic function of a bounded interval by polynomials.)*

5. (20 points) Let K_n be the n^{th} set in the construction of the usual 1/3-Cantor set. This means that $K_0 = [0,1]$, and for each n the set K_{n+1} is a union of closed intervals obtained by deleting the open middle third of each interval from K_n. Also, for each n, let $\mu_n = (\frac{3}{2})^n m|_{K_n}$, meaning that for any Lebesgue measurable set A, $\mu_n(A) = (\frac{3}{2})^n m(A \cap K_n)$.
   (i) Prove that $K = \cap_n K_n$ is compact and non-empty.
   (ii) Prove that $m(K) = 0$.
   (iii) Let $F_n(x) = \mu_n((-\infty, x])$, so F_n is increasing and has $F_n(x) = 0$ for $x \leq 0$ and $F_n(x) = 1$ for $x \geq 1$. Prove that F_n is absolutely continous with respect to Lebesgue measure and find its Radon-Nikodym derivative.
   (iv) Prove that F_n is constant on the intervals in K_n^c and that its value on the j^{th} interval is $j2^{-n}$
   (v) Prove that the sequence $\{F_n(x)\}$ converges uniformly (with respect to x) as $n \to \infty$, to an increasing continuous function $F(x)$ on $[0,1]$.
   (vi) Define μ to be the Lebesgue-Stieltjes measure corresponding to F, so $\mu((a,b]) = F(b) - F(a)$. Prove that μ and m are mutually singular.
   (vii) Prove that $\mu_n(A) \to \mu(A)$ as $n \to \infty$ if A is a closed interval, but that this is not true for arbitrary closed sets.