Abstract Algebra Prelim — Aug. 2016

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let G be a group. Show that if the automorphism group $\mathrm{Aut}(G)$ of G is cyclic then G is abelian. [Hint: Consider the map $G \to \mathrm{Aut}(G)$ given by $g \mapsto \gamma_g$, where $\gamma_g(x) = gxg^{-1}$ for all $x \in G$. What is the kernel of this map?]

2. (a) Let $G = \{x_1, \dots, x_n\}$ be a finite (multiplicative) abelian group of order n. Show that if G has no element of order 2 , then $x_1 \cdot x_2 \cdot \dots \cdot x_n = 1$ and if G has a unique element x of order 2 then $x_1 \cdot x_2 \cdot \dots \cdot x_n = x$.

   (b) For each prime number p, use (a) for a well-chosen G (depending on p) to show that $(p-1)! \equiv -1 \bmod p$.

3. (a) Show every group of order $7^2 \cdot 11^2$ is abelian.

   (b) Use (a) to classify all groups of order $7^2 \cdot 11^2$ up to isomorphism.

4. Let K be a field and let R be the subring of the polynomial ring $K[X]$ given by all polynomials with X-coefficient equal to 0. That is, $R = \{a_0 + a_1 X + a_2 X^2 + \cdots + a_n x^n \in K[X] \mid a_1 = 0\}$.

   (a) Prove X^2 and X^3 are irreducible but not prime in R. You may use that $K[X]$ is a UFD.

   (b) Use (a) to show that the ideal I of R consisting of all polynomials in R with constant term 0 is not principal.

5. Let A be a nonzero ring such that $a^2 = a$ for all $a \in A$. (Examples include $\mathbf{Z}/2\mathbf{Z} \times \cdots \times \mathbf{Z}/2\mathbf{Z}$, but these are not the only ones.)

   (a) Show A has characteristic 2.

   (b) If A is finite, show its size is a power of 2.

   (c) Show every prime ideal in A is maximal.

6. Give examples as requested, with justification.

   (a) An automorphism of S_5 with order 3.

   (b) An irreducible polynomial of degree 10 in $\mathbf{Z}[X]$.

   (c) A field of size 4.

   (d) An infinite field of characteristic 3.