Applied Math Prelim, August 2016

(1) Let $B = \{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H and $M = \text{span } B$. Prove that the closure of M is

$$\{\sum_{n=1}^{\infty} \alpha_n e_n : \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty\}.$$

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au \equiv u'' - u$$

with $u(0) = u(1) = 0$.
(2b) Define $T : L^2(0,1) \to L^2(0,1)$ such that for any $f \in L^2(0,1)$,

$$(Tf)(x) = \int_0^1 G(x,y) f(y)\, dy\ .$$

Explain what spectral theorem is and why it is applicable.
(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3a) Let X and Y be Hilbert spaces and $F : X \to Y$. Give the definition for F being Frechet differentiable at $u_0 \in X$.

(3b) Let $J : C^1[0,1] \to \mathbf{R}$ such that for any $u \in C^1[0,1]$, and any $x \in [0,1]$, we have

$$J(u) \equiv \int_0^1 (\frac{1}{2} u'^2 + \frac{1}{2} u^2 - u)\, dx\ .$$

Show that J is Frechet differentiable at any u. Find such a derivative. In particular show that $J'(u_0) = 0$ when $u_0 = 1$ for all x.

(4a) Let $\{T_j\}_{j=1}^{\infty}$ and S be distribution. Give the definition that $T_j \to S$ in $\mathcal{D}'$, i.e. in the sense of distribution.
(4b) Given a non-negative function $f \in L^1(\mathbf{R}^N)$ with $\int_{\mathbf{R}^N} f\, dx = 1$. Define $g_j(x) = j^N f(jx)$ and let $\tilde{g}_j$ be the distribution induced by g_j. Show that $\tilde{g}_j \to \delta$ in $\mathcal{D}'$.

(4c) Let $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1, & -1 < x < 0, \\ -1, & 0 < x < 1, \\ 1, & 1 < x < 2, \\ 0, & \text{otherwise} \end{cases}$$

Note that it changes sign. Define $\tilde{g}_j$ in in part (b). Is it still true that $\tilde{g}_j \to \delta$ in $\mathcal{D}'$? Justify your claim.

(5) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal sequence in a Hilbert space X. Let

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n + \beta_n \langle x, e_n \rangle e_{n+1},$$

where $\sup |\lambda_n| < \infty$ and $\sup |\beta_n| < \infty$.
(a) Prove that $A : H \to H$ is a linear bounded operator.
(b) Prove A is compact if $\lambda_n \to 0$ and $\beta_n \to 0$.
(c) If A is compact, prove that $\lambda_n \to 0$ and $\beta_n \to 0$. (Hint: consider the sequence $\{Ae_n\}_{n=1}^{\infty}$.)