

# Complex Analysis Prelim

University of Connecticut

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**Instructions:** Do as many of the following problems as you can. Four completely correct solutions will guarantee a PhD pass. A few completely correct solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the complex analysis course, identifying it either by name or stating it in full.

## Notation and Conventions:

- $\mathbb{C}$  denotes the field of complex numbers
- $D(z, r)$  denotes the open disk with center  $z \in \mathbb{C}$  and radius  $r > 0$
- The terminology *analytic function* and *holomorphic function* may be used interchangeably.

## Problems:

1. Prove the *open mapping theorem*: if  $f : U \rightarrow \mathbb{C}$  is a non-constant analytic function defined on a connected, open set  $U \subseteq \mathbb{C}$ , then  $f(V)$  is open for every open set  $V \subseteq U$ .
2. Let  $H := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$  denote the right half plane. Prove that if  $f : H \rightarrow \mathbb{C}$  is analytic and  $f(H) \subseteq D(f(a), r)$  for some  $a \in H$  and  $r > 0$ , then

$$\frac{|f(z) - f(a)|}{|z - a|} \leq \frac{r}{|z + \bar{a}|} \quad \text{for all } z \in H \setminus \{a\}, \quad \text{and} \quad |f'(a)| \leq \frac{r}{2 \operatorname{Re} a}.$$

3. Let  $A := \{z \in \mathbb{C} : r < |z| < R\}$  denote an annulus, where  $0 < r < R$ . Prove that the function  $f(z) := 1/z$  cannot be uniformly approximated in  $A$  by complex polynomials.
4. Let  $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$  be a complex polynomial. Show that there must be at least one point with  $|z| = 1$  and  $|p(z)| \geq 1$ .
5. Let  $f$  be an entire function with the property that for every  $z \in \mathbb{C}$  there is a positive integer  $n = n(z)$  such that  $f^{(n)}(z) = 0$ , where  $f^{(n)}$  denotes the  $n$ -th derivative of  $f$ . Show that  $f$  is a polynomial.
6. Find all 1-1 analytic maps from the upper half disk  $D^+(0, 1) := \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0\}$  onto the unit disk  $D(0, 1)$ .

7. Compute  $\int_0^\infty \frac{x^{1/3}}{x^2 + 1} dx$ . Justify all manipulations. *Hint: Use the contour on the back of this page.*

