Geometry and Topology Exam

August 2016

Instruction: Justify and state your answers in a clear manner. If your answer relies on theorems, please state them.

Problem 1. Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ with the topology given by the basis $\mathcal{B} = \{[a,b) : a < b \text{ and } a, b \in \mathbb{Q}\}$. Determine the closure of the following subsets in $\mathbb{R}_\ell$.

(i) $(1, \sqrt{2})$

(ii) $(\sqrt{2}, 3)$

Problem 2. Let U be an open subset of $\mathbb{R}^2$. Show that U has only countably many connected components.

Problem 3. Let $f : X \to Y$ be a continuous and injective map between topological spaces X and Y. Prove that if X is compact and Y is Hausdorff, then f is an embedding.

Problem 4. Show that there is a two-sheeted covering of the Klein bottle by the torus $\mathbb{T}^2$.

Problem 5. Let A be a subset of a topologic space X. We say that A is a *deformation retract* of X if there is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ and a homotopy between $i \circ r$ and Id_X, where $i : A \to X$ is the inclusion map and Id_X is the identity map on X.

(i) Prove that if A is a deformation retract of X, then the inclusion map i induces an isomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ for any base point $a \in A$.

(ii) Use part (i) to give the fundamental group of each of the following spaces. (Please describe an appropriate deformation retract, using pictures, words, or both, but you do not need to prove that it is a deformation retract.)

(1) $\mathbb{R}^2$ with the origin removed.

(2) The torus $S^1 \times S^1$ with one point removed.

Problem 6. Let X be a topological space obtained by taking two copies of real projective plane P^2 and identifying a single point p in one copy with a single point q in another copy. Determine the fundamental group of X.