Probability Prelim Exam for Actuarial Students
August 2016

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

1. (10 points) For $t \in [0,1]$, we define $X_t = B_t - tB_1$, where $\{B_t : t \geq 0\}$ is a standard Brownian motion.
   (a) (5 points) Find the density of X_t.
   (b) (5 points) Show that the density of X_t is equal to the limit of: the density of B_t conditioned on $|B_1| < 1/n$, as $n \to \infty$.

2. (20 points) A monkey is typing randomly the keys in a keyboard with only 10 digits: 0,1,2,...,9. Let the sequence of digits typed by the monkey be denoted by
$$r_1, r_2, \ldots.$$
   Assume that the monkey types each of the ten digits with equal chance.
   (a) (10 points) For $n = 1, 2, \ldots$, at time n a new player appears and bets \$1 on that $r_n = 2$. If the player wins, he gets \$10 (\$9 award plus the original \$1) and then bets the \$10 on that $r_{n+1} = 0$. If the player wins again, he gets \$100 (\$90 award plus the original \$10) and then bets the \$100 on $r_{n+2} = 1$. If the player wins again, he gets \$1,000 (\$900 award plus the original \$100) and then bets the \$1,000 on that $r_{n+3} = 6$. If the player wins again, he gets \$10,000 (\$9,000 award plus the original \$1,000) and stops betting. If the player loses at any time, he stops betting. Let S_n be the total amount won by all players by time n and $S_0 = 0$. Show that $\{S_n : n \geq 0\}$ is a martingale.
   (b) (10 points) Let τ be the first time when the monkey types the sequence "2016." Compute $E[\tau]$.

3. (10 points) Let X be a real-valued random variable. Suppose that
$$E\left[3^X\right] = 9.$$
   Show that
$$P(X \geq 3) \leq \frac{1}{3}.$$

4. (10 points) Let $\delta > 0$. Let $X_1, X_2, \ldots$ be a sequence of independent non-negative random variables such that
$$P(X_i \geq \delta) \geq \frac{1}{i}, \quad i = 1, 2, \ldots.$$
   Show that $\sum_{i=1}^{\infty} X_i = \infty$ with probability one.

5. (10 points) Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each having the same mean m and each having the same variance $\leq v < \infty$. Let $\epsilon > 0$. Show that

$$\lim_{n\to\infty} P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - m\right| \geq \epsilon\right) = 0$$

6. (10 points) Let $(\Omega, \mathscr{F}, P)$ be a probability space, where $\Omega = \{1, 2, 3\}$, $\mathscr{F}$ is the set of all subsets of Ω, and $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. Let X and Y be random variables defined on $(\Omega, \mathscr{F}, P)$ as follows:

$$X(1) = 5, \quad X(2) = 5, \quad X(3) = 6$$

and

$$Y(1) = 1, \quad Y(2) = 2, \quad Y(3) = 3.$$

(a) (3 points) Calculate $\sigma(X)$ precisely, where $\sigma(X)$ is the σ-algebra generated by X.

(b) (2 points) State the definition of $E[Y|X]$.

(c) (5 points) Let $Z = E[Y|X]$. Calculate (with proof) $Z(\omega)$ for each $\omega \in \Omega$.