Real Analysis Preliminary Exam, August 2016

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by m or dx. The σ-algebra of the Lebesgue measurable sets in $\mathbb{R}^n$ is denoted by $\mathcal{M}_n$. If A is a set then the set of all subsets of A is denoted by $\mathcal{P}(A)$.

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1. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space.
   (a) Define the notions of μ-a.e. convergence and convergence in measure.
   (b) If $(X, \mathcal{A}, \mu)$ is a finite measure space prove that μ-a.e. convergence implies convergence in measure.
   (c) Prove that (b) might fail if $(X, \mathcal{A}, \mu)$ is an infinite measure space.

2. (10 points) Prove or disprove the following statements.
   (a) Hyperplanes in $\mathbb{R}^n$ have infinite n-dimensional Lebesgue measure.
   (b) $\mathcal{M}_1 \times \mathcal{M}_1 \neq \mathcal{M}_2$.
   (c) For $0 < p \leq \infty$ the equivalence class of an $f \in L^p(\mathbb{R}, m)$ contains at most one continuous function.

3. (10 points) Compute
$$\lim_{j\to\infty} \int_{-j}^{j} \frac{\sin(x^j)}{x^{j-2}}\, dx, \;\; j \in \mathbb{N},$$
   and provide full justification for all steps in your reasoning.

4. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. Prove that there exists a Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ such that $g = f$, m-a.e.

5. (10 points) Let μ be the counting measure in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Show that if $1 \leq p < q \leq \infty$, then
$$\|f\|_{L^\infty(\mu)} \leq \|f\|_{L^q(\mu)} \leq \|f\|_{L^p(\mu)} \leq \|f\|_{L^1(\mu)}.$$

6. (10 points) A set $A \subset \mathbb{R}^n$ is called *porous* if for all $x \in A$ there exists some $\delta_x \in (0, 1)$ and two sequences $r_i > 0, y_i \in \mathbb{R}^n, i \in \mathbb{N}$, (both depending on x) such that $r_i \to 0$ and
$$B(y_i, \delta_x r_i) \subset B(x, r_i) \setminus A.$$
   (a) Give an example of an uncountable porous set.
   (b) Show that a Lebesgue measurable porous set has Lebesgue measure zero.