Abstract Algebra Prelim Aug. 2017

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let G be a nontrivial finite group. A subgroup M is called a *maximal* subgroup of G if M is a proper subgroup, *i.e.*, $M \neq G$, and the only subgroups of G containing M are M and G.
   (a) Show each proper subgroup of a finite group G is contained in a maximal subgroup of G.
   (b) Count the number of maximal subgroups of the dihedral group of order $2p$, where p is an odd prime.
   (c) Show that if a nontrivial finite group G has only one maximal subgroup then G is cyclic of prime-power order. (Hint: first prove G is cyclic.)

2. Let G be a nonabelian group of order 75 and H be a 5-Sylow subgroup of G.
   (a) Show H is a normal subgroup of G and is abelian.
   (b) Show H is not cyclic, or equivalently $H \cong (\mathbf{Z}/5\mathbf{Z})^2$. (Hint: show the conjugation action of G on H is not trivial.)
   (c) Determine a 2×2 matrix A with entries in $\mathbf{Z}/5\mathbf{Z}$ that has order 3. (Hint: you can find such a matrix with integer entries having *complex eigenvalues* equal to the primitive 3rd roots of unity ζ_3 and ζ_3^2, where $\zeta_3 = \frac{-1+\sqrt{-3}}{2}$.)
   (d) Construct an example of a nonabelian group with order 75. (The matrix in part (c) will be useful.)

3. The group S_9 denotes the permutations of a set with 9 elements (symmetric group).
   (a) Prove that there is no element of order 18 in S_9.
   (b) Construct, with justification, an element of order 20 in S_9.

4. (a) Prove the only units in $\mathbf{Z}[\sqrt{-5}]$ are ± 1.
   (b) Justify why the equation $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ shows $\mathbf{Z}[\sqrt{-5}]$ is not a unique factorization domain.
   (c) Justify why the equation $(2+3i)(2-3i) = (3+2i)(3-2i)$ does **not** show that $\mathbf{Z}[i]$ is not a unique factorization domain.

5. Let R be a ring with identity, possibly noncommutative, and let M be a left R-module. Denote by $\mathrm{End}(M)$ the set of R-module endomorphisms of M, that is,
   $$\mathrm{End}(M) = \{f \colon M \to M \mid f \text{ is an } R\text{-module homomorphism}\}.$$
   (a) Prove that $\mathrm{End}(M)$ is a ring with identity under the operations
   $$(f+g)(m) = f(m) + g(m) \text{ and } (fg)(m) = (f \circ g)(m) \text{ where } f, g \in \mathrm{End}(M), m \in M.$$
   (b) Suppose R is a commutative ring with identity and $M = R$. Show the ring $\mathrm{End}(R)$ is commutative.

6. Give examples as requested, with justification.
   (a) Two nonabelian groups of order 12 that are not isomorphic.
   (b) A group that acts transitively on the plane minus the origin, $\mathbf{R}^2 - \{\binom{0}{0}\}$.
   (c) A ring that is not a field and has infinitely many units.
   (d) A nonzero prime ideal in $\mathbf{R}[x, y]$ that is not a maximal ideal.