Applied Math Prelim August 2017

1. Let $\tau_0 \in (0,1)$.

   (a) (10 pts) Find Green's function for

   $$\begin{cases} y'' + y = \delta(x - \tau_0) \\ y'(0) = y(1) = 0 \end{cases}$$

   (b) (10 pts) Show that there exists a unique solution for

   $$\begin{cases} y'' + y = \lambda \tan^{-1} y + \cos x \\ y'(0) = y(1) = 0 \end{cases}$$

   for $|\lambda|$ sufficiently small.

2. (20 pts) Show that the unit ball in a normed linear space is compact iff the space is finite dimensional.

3. (20 pts) Prove that a linear transformation between normed linear space is continuous iff it maps some nonvoid open set in the domain space into a bounded set in the range space.

4. (15 pts) If A is a compact operator on a Banach space, then the range of $I - A$ is closed.

5. (15 pts) Let f be a differentiable map between normed linear space. Let y_0 be a point such that f' is invertible at every point in $f^{-1}(y_0)$. Prove $f^{-1}(y_0)$ is a discrete set.

6. (10 pts) Let $\{x_i\}$ be a list of all rational points in $\mathbb{R}^n$. Define T by $T(\varphi) = \sum_{i=1}^{\infty} 2^{-i} \varphi(x_i)$ for any test function φ. Prove T is a distribution.