Complex Analysis Prelim

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Instructions: Do as many of the following problems as you can. Four completely correct solutions will guarantee a PhD pass. A few completely correct solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the complex analysis course, identifying it either by name or stating it in full.

Notation: $\mathbb{C}$ is the complex plane and $\mathbb{D}$ is the open unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Suppose that f is an entire function satisfying $f(n) = n$ for $n = 1, 2, \ldots$ and $\lim_{|z|\to\infty} |f(z)| = \infty$. Show that $f(z) = z$.

2. Let f be an analytic function on an open set containing the closure of $\mathbb{D}$, except for a simple pole at z_0 with $|z_0| = 1$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series for f in $\mathbb{D}$. Show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

3. Show that there does not exist an analytic $f : \mathbb{D} \to \mathbb{C}$ satisfying $\lim_{|z|\to 1} |f(z)| = \infty$.

4. Let f be an entire function satisfying that $\mathrm{Re}\, f - 4\mathrm{Im}\, f$ is bounded. Show that f is constant.

5. Find a conformal mapping from $\mathbb{D}$ onto open set bounded between
$$\{z : |z| = 1\} \text{ and } \{z : |z - 1| = 2\}.$$

6. Let $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$ be a complex polynomial, and let
$$R = \max(1, |c_0| + |c_1| + \cdots + |c_{n-1}|).$$
Show that all the roots of p are in $\{z : |z| \leq R\}$.

7. Use contour integration to show
$$\int_0^{\infty} \frac{\sin^2 u}{u^2} du = \frac{\pi}{2}.$$
(Hint: show first that for real z, $\sin^2 z = \mathrm{Re}\frac{1-e^{2iz}}{2}$).